MATH 410 : Supplementary Exercises

This file containing interesting problems from outside the textbook will be updated throughout the semester. Most of these problems will be assigned in the homework.

1. Use the Well-Ordering Principle to show that $\sqrt{2}$ is irrational.

2. Prove that the expression $(3^{3n+3} - 26n - 27)$ is a multiple of 169 for all $n \in \mathbb{N}$.

3. Let $x \neq 1$ be any real number. Then prove that $\sum_{j=0}^{n-1} x^j = \frac{x^n - 1}{x-1}$.

4. Prove that if a and b are odd integers, then $a^2 - b^2$ is divisible by 8.

5. Prove that for all $n \in \mathbb{N}$, $(1+\sqrt{2})^{2n} + (1-\sqrt{2})^{2n}$ is an even integer and $(1+\sqrt{2})^{2n} - (1-\sqrt{2})^{2n}$ equals $b\sqrt{2}$ for some integer b.
(*Hint:* Prove both the statements simultaneously using induction on n.)

6. Prove that any two consecutive Fibonacci numbers are relatively prime.
(Fibonacci numbers are defined as: $f_0 = 1$, $f_1 = 1$, $f_{n+1} = f_n + f_{n-1}$ for $n \geq 1$.)

7. Use the textbook exercise 2.3.20f (solved in HW#2) to show that $gcd(a^2, b^2) = (gcd(a, b))^2$.

8. Let a, b, c, d be positive integers with $b \neq d$. Prove that: If $gcd(a, b) = 1$ and $gcd(c, d) = 1$ then $\frac{a}{b} + \frac{c}{d}$ is not an integer.

9. Find the smallest positive integer n such that the Diophantine equation $10x + 11y = n$ has exactly nine non-negative solutions.

10. What is the smallest positive rational number that can be expressed in the form $\frac{x}{30} + \frac{y}{36}$ with $x, y \in \mathbb{Z}$?

11. Prove that: If $2^n - 1$ is prime then n is prime. [Compare this to #3.1.11b in the textbook.]

12. Let $F_n = 2^{2^n} + 1$, $n \geq 0$ (these are called Fermat numbers). Show that $gcd(F_m, F_n) = 1$ for $m > n \geq 0$.

13. Prove there are infinitely many primes of the form $4k + 1$.

14. Find the smallest integer divisible by 2 and 3 which is simultaneously a square and a fifth power.

15. Prove that

$$\frac{(lcm(a,b,c))^2}{lcm(a,b)\ lcm(b,c)\ lcm(c,a)} = \frac{(gcd(a,b,c))^2}{gcd(a,b)\ gcd(b,c)\ gcd(c,a)}$$

16. If p and $p+2$ are twin primes, with $p > 3$, then prove that $6|\ (p+1)$.

17. Let $N = (a_m a_{m-1} \ldots a_2 a_1 a_0)_{10}$. Let $M = a_m 10^{m-1} + \ldots + a_3 10^2 + a_2 10^1 + a_1$. Then show that

(a) $7|N \Leftrightarrow 7|(M - 2a_0)$.

(b) $13|N \Leftrightarrow 13|(M - 9a_0)$.

[Note that the repeated application of these criteria on a number gives an efficient procedure to check for divisibility by 7 or 13.]

18. Find one million consecutive integers that are NOT square-free. [Note that n is not square-free iff $p^2|n$ for some prime p, i.e., square (or a higher power) of a prime occurs in its prime factorization.]

19. The converse to the Fermat's little theorem is true in the following sense:
Show that: If $n \geq 2$ and for all a, $1 \leq a \leq n-1$, $a^{n-1} \equiv 1(mod\ \ n)$, then n must be prime.

20. Observe that $1 + \frac{1}{3} = \frac{4}{3}$, $1 + \frac{1}{2} + \frac{1}{4} = \frac{7}{4}$, $1 + \frac{1}{5} = \frac{6}{5}$, $1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{6} = \frac{12}{6}$, $1 + \frac{1}{7} = \frac{8}{7}$, $1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} = \frac{15}{8}$, and so on. Conjecture a theorem that implies all these observations, and then prove that theorem.

21. Prove that: If $f(n) = \prod_{d|n} g(d)$ then $g(n) = \prod_{d|n} (f(d))^{\mu(\frac{n}{d})}$.

22. Prove that $\frac{\sigma(n)}{\tau(n)} \geq \sqrt{n}$. [Hint: First show that $\frac{\sigma(n)}{\tau(n)} \geq \prod_{d|n} d^{\frac{1}{\tau(n)}}$.]

23. If $n-1$ and $n+1$ are twin primes with $n > 4$ then show that $\phi(n) \leq \frac{n}{3}$.

24. For a fixed positive integer k, if $\phi(x) = k$ has a unique integer solution, say $x = n_0$, then show that $36|n_0$. [Comment: compare this to Exercise 17b in Section 7.2].

25. Observe that: $1+2 = \frac{3 \cdot 2}{2}$; $1+3 = \frac{4 \cdot 2}{2}$; $1+2+3+4 = \frac{5 \cdot 4}{2}$; $1+5 = \frac{6 \cdot 2}{2}$; $1+2+3+4+5+6 = \frac{7 \cdot 6}{2}$; $1+3+5+7 = \frac{8 \cdot 4}{2}$, and so on. Conjecture a theorem that implies all these observations, and then prove that theorem.

26.(a) First prove 8.1.6c from the textbook and then use it to prove that: there are infinitely many primes of the form $6k+1$.
(b) First prove 8.1.8a from the textbook and then use it to prove 8.1.9 from the textbook.

27. Prove that the product of all the primitive roots of a prime $p > 3$ is congruent to 1 modulo p.

28. Let p be an odd prime with $gcd(a, p) = 1$. Let $d = gcd(m, p-1)$. Prove that:
$x^m \equiv a \pmod{p}$ has a solution if and only if $a^{(p-1)/d} \equiv 1 \pmod{p}$. [Comment: This is a generalization of Euler's Criterion for mth-power residues of an odd prime]