MATH 554 : Homework #1

Submit five out of the following six problems. Problems 1, 2, and 3 are compulsory.
Due Thursday, February 2nd.

Before starting the HW, carefully read the HW related discussion and rules described in
http://www.math.iit.edu/~kaul/TeachingSpr23/TeachingFiles/Math554.pdf

Always remember that homework is NOT meant to be an examination, it is meant to assist in your learning and development. If you are having trouble understanding a homework problem, I will be glad to direct you in the right direction without giving away the solution. You are encouraged to ask questions during the *Class*, through the *Campuswire Discussion Forums*, during the *Office Hours*, or through *Email to me*.

All problems require explicit and detailed proofs/ arguments/ reasons. Solutions should be written clearly, legibly, and concisely, and will be graded for both mathematical correctness and presentation. Points will be deducted for sloppiness, incoherent or insufficient explanation, or for lack of supporting rationale.

You are allowed to discuss the homework problems with no one except your classmates, and the instructor. However, the solutions should be written by you and you alone in your own words. Any incident of plagiarism/ cheating (from a person or from any online resource) will be strictly dealt with.

If you discuss the problems with anyone, you are required to note their name at the top of your HW submission under a subtitle "Collaborator" or "Discussed with:".

1. Prove that if a graph G with n vertices and m edges has a matching with k edges, then G has a bipartite subgraph with at least $(m+k)/2$ edges.

2. (a) Prove that if there exists a $p \in (0,1)$ such that $\binom{n}{k}p^{\binom{k}{2}} + \binom{n}{l}(1-p)^{\binom{l}{2}} < 1$, then $R(k,l) > n$.
(b) Prove that $R(k,l) > n - (\binom{n}{k}p^{\binom{k}{2}} + \binom{n}{l}(1-p)^{\binom{l}{2}})$ for all $n \in \mathbb{N}$ and $p \in (0,1)$. Choose n and p in this bound (in terms of l) to prove $R(3,l) > l^{3/2-o(1)}$, where $o(1)$ represents a function of l that tends to zero as l tends to infinity.

3. In list coloring of a graph G, each vertex is assigned a list of colors $L(v)$, which are the only colors allowed to be used on v in a proper coloring of G. G is said to be *k-choosable* if it has a proper coloring no matter which list $l(v)$ of k colors is assigned to each vertex v. $\chi_\ell(G)$, the *list chromatic number* of G, is the smallest k such that G is k-choosable.

(a) Let $G = (V,E)$ be a bipartite graph on n vertices. Show that $\chi_\ell(G) \leq 1 + \log_2 n$. That is, for any list assignment of G with a list $L(v)$ of at least $1 + \log_2 n$ colors associated with each vertex $v \in V$, prove that there is a proper coloring of G assigning to each vertex v a color from its list $L(v)$.

(b) If there exists a non-2-colorable k-uniform hypergraph with n edges then prove that $K_{n,n}$ is not k-choosable.

(c) Use parts (a) and (b) to prove closely matching lower and upper bounds on $\chi_\ell(K_{n,n})$.

4. The Van der Waerden number $w(l,k)$ is defined as the least n such that every k-coloring of $[n] = \{1, \dots n\}$ has a monochromatic l-term arithmetic progression. Prove that $w(l,k) > (lk^{l-1})^{1/2}$.

5. Prove that every graph G with n vertices and m edges has a bipartite subgraph with at least $m\frac{\lceil n/2 \rceil}{2\lceil n/2 \rceil - 1}$ edges.

6. United Federation of Planets has 1600 administrators, who have formed 16000 committees of 80 members each. Prove that there must be two committees with at least four common members.