MATH 554 : Homework #3

Attempt at least four of the following five problems, with Problem 1 as compulsory.
Solving a fifth problem will count as a bonus.
Due Thursday, March 2nd.

Before starting the HW, carefully read the HW related discussion and rules described in `http://www.math.iit.edu/~kaul/TeachingSpr23/TeachingFiles/Math554.pdf`

Always remember that homework is NOT meant to be an examination, it is meant to assist in your learning and development. If you are having trouble understanding a homework problem, I will be glad to direct you in the right direction without giving away the solution. You are encouraged to ask questions during the *Class*, through the *Campuswire Discussion Forums*, during the *Office Hours*, or through *Email to me*.

All problems require explicit and detailed proofs/ arguments/ reasons. Solutions should be written clearly, legibly, and concisely, and will be graded for both mathematical correctness and presentation. Points will be deducted for sloppiness, incoherent or insufficient explanation, or for lack of supporting rationale.

You are allowed to discuss the homework problems with no one except your classmates, and the instructor. However, the solutions should be written by you and you alone in your own words. Any incident of plagiarism/ cheating (from a person or from any online resource) will be strictly dealt with.

If you discuss the problems with anyone, you are required to note their name at the top of your HW submission under a subtitle "Collaborator" or "Discussed with:".

1. In the binomial random model, show that $t(n) = \frac{\log n}{n}$ is a threshold function for the disappearance of isolated vertices.
   (Hint: The following fact might be useful: if $np^2 \to 0$ as $n \to \infty$, then $(1-p)^n \sim e^{-np}$. Also, it is easier to work with $t(n) = c\frac{\log n}{n}$, and set $c > 1$ for one part of the threshold calculation and $c < 1$ for the other part.)

2. A random labeled tournament is generated by orienting each edge v_iv_j $(i < j)$ as $v_i \to v_j$ or $v_i \leftarrow v_j$ independently with probability $1/2$.

   (a) In a tournament, a king is a vertex such that every other vertex can be reached from it by a path of length at most 2. It is known that every tournament contains a king. Is it true that in almost every tournament every vertex is a king?

   (b) Prove that almost every tournament is strongly connected[1].

[1] A directed graph is said to be *strongly connected* if for every ordered pair of vertices (u, v) there is directed path from u to v.

3. Let Q_k be the following graph property: for every choice of disjoint vertex sets S, T of size k, there is an edge with endpoints in S and T. For $k = c \log_2 n$, prove that almost every graph from the binomial random graph space $G(n, 1/2)$ has property Q_k if $c > 2$.

4. Prove that the length of the longest constant run in a list of n random heads and tails (of a fair coin) is $(1 + o(1)) \log_2 n$. In other words, for $k = (1 + \varepsilon) \log_2 n$, almost no list has k consecutive identical flips if $\varepsilon > 0$, and almost every list has k consecutive identical flips if $\varepsilon < 0$.

5. Prove that if $k = \log_2 n - (2 + \varepsilon) \log_2 \log_2 n$, then almost every n-vertex tournament has the property that every set of k vertices has a common successor.