MATH 554 : Homework #5

Submit four out of the five problems below.
Due Thursday, March 30th.

Before starting the HW, carefully read the HW related discussion and rules described in
http://www.math.iit.edu/~kaul/TeachingSpr23/TeachingFiles/Math554.pdf

Always remember that homework is NOT meant to be an examination, it is meant to assist in your learning and development. If you are having trouble understanding a homework problem, I will be glad to direct you in the right direction without giving away the solution. You are encouraged to ask questions during the *Class*, through the *Campuswire Discussion Forums*, during the *Office Hours*, or through *Email to me*.

All problems require explicit and detailed proofs/ arguments/ reasons. Solutions should be written clearly, legibly, and concisely, and will be graded for both mathematical correctness and presentation. Points will be deducted for sloppiness, incoherent or insufficient explanation, or for lack of supporting rationale.

You are allowed to discuss the homework problems with no one except your classmates, and the instructor. However, the solutions should be written by you and you alone in your own words. Any incident of plagiarism/ cheating (from a person or from any online resource) will be strictly dealt with.

If you discuss the problems with anyone, you are required to note their name at the top of your HW submission under a subtitle "Collaborator" or "Discussed with:".

1. In class we used Entropy to prove that $n^2 \leq n_1 n_2 n_3$ where n is the number of distinct points in $\mathbb{R}^3$ which have n_1, n_2, n_3 distinct projections on the three standard planes, respectively. State and prove (using Entropy) a generalization of this result for points in $\mathbb{R}^d$. Is your bound sharp for all d? [Comment: Many different generalizations might be possible based on which projections you wish to consider.]

2. Define $[n] = \{0, \ldots, n-1\}$. Fix $k \in [n]$ such that k divides n. Let $\mathcal{F} \subseteq 2^{[n]}$, a family of subsets of $[n]$, such that $\forall F, F' \in \mathcal{F}$, $\exists i \in [n-k]$ such that $\{i+j \mid 0 \leq j \leq k-1\} \subseteq F \cap F'$. State and prove a sharp upper bound on $|\mathcal{F}|$.

3. (a) Let $G = (A, B; E)$ be a bipartite graph with $|A| = |B| = n$. Then prove that the number of perfect matchings in G is at most $\prod_{v \in A} d(v)$, where $d(v)$ is the degree of vertex v.

   (b) The famous Bregman's bound states that the number of perfect matchings in G is at most $\prod_{v \in A} (d(v)!)^{1/d(v)}$. As an optional problem try proving this. Which part of your proof of (a) needs to be improved to get Bregman's bound? There is a short Entropy based proof of Bregman's bound, ask me for the reference.

4. (a) Suppose $G_1, \ldots, G_t$ be bipartite graphs with the same vertex set $[n]$ such that union of their edge sets equals K_n, that is K_n is decomposed into t bipartite graphs. The prove that $t \geq \log n$. [Comment: A short proof using properties of chromatic number is possible (that many of you have done in Math 553) but I am looking for an entropy based proof.]

   (b) State and prove a generalization where G_i are all k-partite graphs. [Comment: Fredman and Komlos generalize this result to decomposition of a complete r-uniform hypergraph on n vertices into t k-partite r-uniform hypergraphs.]

5. Let k be a positive integer. Let Ω be a finite set and $\mathcal{S} = \{S_1, \ldots, S_m\}$ be a collection of subsets of Ω such that each element of Ω is contained in at least k members of $\mathcal{S}$. Let $\mathcal{F}$ be a collection of subsets of Ω. For each $f \in \mathcal{F}$, denote $f \cap S_i$ by f_i and let $\mathcal{F}_i = \{f_i | f \in \mathcal{F}\}$. Let every such set f_i be endowed with a positive integer weight $w_i(f_i)$. Then prove that

$$\left(\sum_{f \in \mathcal{F}} \prod_{i=1}^{m} w_i(f_i) \right)^k \leq \prod_{i=1}^{m} \sum_{f_i \in \mathcal{F}_i} (w_i(f_i))^k$$

   [Note that setting all the weights equal to 1 gives us the corollary to Shearer's lemma that we proved in class. An easy generalization of this result can be used to prove classical inequalities like Cauchy-Schwarz, Holder, AM-GM, and many more.]