MATH 554 : Homework #7

Submit the first three problems below. The last two problems are optional.
Due Thursday, April 27th.

Before starting the HW, carefully read the HW related discussion and rules described in
http://www.math.iit.edu/~kaul/TeachingSpr23/TeachingFiles/Math554.pdf

Always remember that homework is NOT meant to be an examination, it is meant to assist in your learning and development. If you are having trouble understanding a homework problem, I will be glad to direct you in the right direction without giving away the solution. You are encouraged to ask questions during the *Class*, through the *Campuswire Discussion Forums*, during the *Office Hours*, or through *Email to me*.

All problems require explicit and detailed proofs/ arguments/ reasons. Solutions should be written clearly, legibly, and concisely, and will be graded for both mathematical correctness and presentation. Points will be deducted for sloppiness, incoherent or insufficient explanation, or for lack of supporting rationale.

You are allowed to discuss the homework problems with no one except your classmates, and the instructor. However, the solutions should be written by you and you alone in your own words. Any incident of plagiarism/ cheating (from a person or from any online resource) will be strictly dealt with.

If you discuss the problems with anyone, you are required to note their name at the top of your HW submission under a subtitle "Collaborator" or "Discussed with:".

1. Prove that the minimum number of hyperplanes in $\mathbb{R}^n$ that do not contain the origin but together cover all the other points in $\{0,1\}^n$ (corners of the unit hypercube) is n. [Comment: For some fixed $a \in \mathbb{R}^n$ and $b \in \mathbb{R}$, a hyperplane consists of all $x \in \mathbb{R}^n$ such that $a^T x = b$].

2. For each vertex v in graph G, specify $B(v) \subseteq \{1, \ldots, d_G(v)\}$, where $d_G(v)$ is the degree of v in G. Prove that if $\sum_{v \in V(G)} |B(v)| < |E(G)|$, then G has a non-trivial subgraph H such that $d_H(v) \notin B(v)$ for all $v \in V(G)$ (note: non-trivial means not all vertices can have degree zero in H).

3. Let p be a prime and $(a_1, b_1), \ldots, (a_{2p-1}, b_{2p-1})$ be a sequence of integers such that $a_i, b_i \in \mathbb{Z}_p$, $\forall i$. Prove that there exists a non-empty set $I \subseteq \{1, \ldots, 2p-1\}$ such that $\sum_{i \in I}(a_i, b_i) = (0, 0)$ modulo p.

4. Given an odd prime p, and integer k, $1 \leq k < p$. Consider arbitrary elements $a_1, \ldots, a_k \in \mathbb{F}_p$, and distinct elements $b_1, \ldots, b_k \in \mathbb{F}_p$. Prove that there is a permutation σ of $[k]$ such that for $1 \leq i \leq k$ the values $a_i + b_{\sigma(i)}$ are distinct modulo p.

5. Prove that $\chi_\ell(C_{2k+1} \square P_2) = 3$ for any $k \geq 1$. Can you extend the argument to find the list chromatic number of $C_{2k+1} \square P_n$?