MATH 454: Homework #1

Due Wednesday, 1/17, before 11pm via a PDF file uploaded to the Homework#1 under Assignments in the Blackboard course page.

You are allowed to discuss the homework problems with no one except your classmates, the TA, and the instructor. However, the solutions should be written by you and you alone in your own words. **If you discussed HW problems with a classmate or TA, you have to write their name at the top of the HW submission as a collaborator.** Any incident of plagiarism/ cheating (from a person or from any online resource) will be strictly dealt with.

Re-read the "HW Discussion and Solution Rules" and " 'Why and How' of Homework" sections of the course information sheet for some important advice on the HWs for this course.

All problems require explicit and detailed explanations. Solutions should be written clearly, legibly, and concisely, and will be graded for both mathematical correctness and presentation. Points will be deducted for sloppiness, incoherent or insufficient explanation, or for lack of supporting rationale.

Always remember that homework is NOT meant to be an examination, it is meant to assist in your learning and development. If you need help with any HW problem, don't hesitate to ask me. You are encouraged to ask questions during my *Office Hours*, during the *TA office hours*, or through *Email to me*.

Submit a total of 5 problems out of the 6 listed below. Problems 5 and 6 are compulsory.

1. Let $\Delta(G)$ denote the maximum degree, and $\delta(G)$ denote the minimum degree of vertices in a graph G.

Given a graph G and a subgraph H of G, consider the following two statements:
(1) $\delta(G) \geq \delta(H)$ and (2) $\Delta(G) \geq \Delta(H)$.

(a) Only one of these statements is true for all graphs. Which is it? Given a short proof of the truth of that statement.
(b) Find a counterexample showing the other statement is not always true.

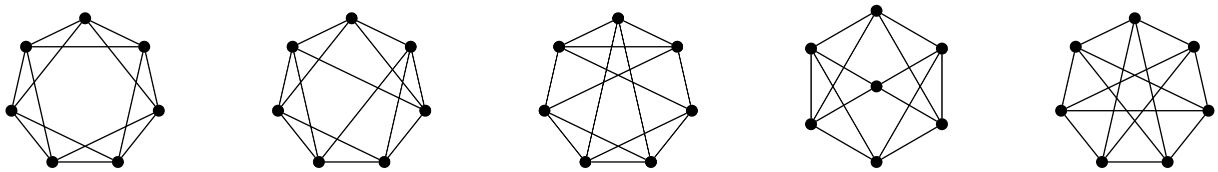
2. Prove that every set of six people contains at least three mutual acquaintances or three mutual strangers. As the first step, first show how to convert this statement into a graph-theoretic statement (as discussed in class), and then prove that graph-theoretic statement.

3. Let $N(n, k)$ denote the number of nonisomorphic simple graphs with n vertices and k edges. Compute $N(4, 3)$ by explicitly finding all such graphs.

Find all nonisomorphic simple graphs of order 4, by finding all $N(4, k)$ as above.

Comment: Remember that you have justify your claims and explain how you have considered all possibilities for graphs on 4 vertices.

4. Which pairs of graphs below are isomorphic (and which are non-isomorphic)? Remember you have justify your answer.



5. If G_1 and G_2 are complementary graphs (that is, they are complements of each other, $\overline{G_1} = G_2$) then prove that at least one of them must be connected.

6. Prove that a self-complementary graph with n vertices exists if and only if n or $n-1$ is divisible by 4 (that is $n = 4k$ or $n = 4k+1$ for some k). Recall that G is self-complementary means that G is isomorphic to its own complement, $\overline{G} = G$.
Comment: Note that this is an 'if and only if' statement which means you have to prove two implications - forward and backward. The forward implication asks you to prove that a self-complementary graph can only have a certain number of vertices. The backward implication is asking you to give a construction of a self-complementary graph on n vertices when n satisfies the divisibility condition. You have describe (use figures if you like) how such a graph on any number of vertices (not just specific small values of n) can be constructed. As a hint, start with P_4 and try to generalize it.