<u>MATH 454: Homework #4</u>

Due Wednesday, 2/7, before 11:59pm via a single PDF file uploaded to the Homework#4 under Assignments in the Blackboard course page.

You are allowed to discuss the homework problems with no one except your classmates, the TA, and the instructor. However, the solutions should be written by you and you alone in your own words. **If you discussed HW problems with a classmate or TA, you have to write their name at the top of the HW submission as a collaborator.** Any incident of plagiarism/ cheating (from a person or from any online resource) will be strictly dealt with.

Re-read the "HW Discussion and Solution Rules" and " 'Why and How' of Homework" sections of the course information sheet for some important advice on the HWs for this course.

All problems require explicit and detailed explanations. Solutions should be written clearly, legibly, and concisely, and will be graded for both mathematical correctness and presentation. Points will be deducted for sloppiness, incoherent or insufficient explanation, or for lack of supporting rationale.

Always remember that homework is NOT meant to be an examination, it is meant to assist in your learning and development. If you need help with any HW problem, don't hesitate to ask me. You are encouraged to ask questions during my *Office Hours*, during the *TA office hours*, or through *Email to me*.

Submit Problem 1, and any four out of the remaining five problems stated below. Be sure to complete the reading HW before you attempt these problems.

1. Prove or disprove each of the following statements:
(a) There exist infinitely many connected Eulerian graphs of even order (# of vertices) and odd size (# of edges).
(b) Every Eulerian bipartite graph has an even number of edges.
(c) Every maximal trail in an even graph is an Eulerian trail.

2. Solve each of the following (unrelated) problems:
(a) Suppose in a bipartite graph all the vertices except one have the same degree d, and the remaining vertex has degree x. Show that x must be a multiple of d.
(b) Define a simple graph R_k on vertex set of Q_k by making two binary k-tuples adjacent iff they agree in exactly one coordinate. Prove that R_k is isomorphic to Q_k if and only if k is even.

3. Show that each edge in the Petersen graph belongs to exactly four 5-cycles. Use this to show that the Petersen Graph has exactly twelve 5-cycles. What are the maximum number of edges in a bipartite subgraph of the Petersen graph?
<u>Comment</u>: Students who have taken Math 453 are especially encouraged to think about this problem and also the textbook problem 1.3.32.

4. Let G be a graph on n vertices, $n \geq 2$. Determine the **maximum** possible number of edges in G under each of the following conditions (separately):

(a) G has an independent set of size a.

(b) G has exactly k components.

5. Let G be a simple graph with average degree $a > 0$.

(a) Prove that $G - v$ has average degree at least a if and only if $d(v) \leq a/2$.

(b) Use part (a) to give an algorithmic proof that G has a subgraph with minimum degree greater than $a/2$.

<u>Comment</u>: In an algorithmic proof, you describe a step-by-step process (an algorithm), along with a proof that the process culminates in the answer you are looking for.

6. Let G be a connected graph with exactly $2k$ odd degree vertices, where $k \geq 1$. Use induction on k to prove that G can be decomposed using exactly k trails.