MATH 454: Homework #5

Due Thursday, 2/15, before 11am via a single PDF file uploaded to the Homework#5 under Assignments in the Blackboard course page.

You are allowed to discuss the homework problems with no one except your classmates, the TA, and the instructor. However, the solutions should be written by you and you alone in your own words. **If you discussed HW problems with a classmate or TA, you have to write their name at the top of the HW submission as a collaborator.** Any incident of plagiarism/ cheating (from a person or from any online resource) will be strictly dealt with.

Re-read the "HW Discussion and Solution Rules" and " 'Why and How' of Homework" sections of the course information sheet for some important advice on the HWs for this course.

All problems require explicit and detailed explanations. Solutions should be written clearly, legibly, and concisely, and will be graded for both mathematical correctness and presentation. Points will be deducted for sloppiness, incoherent or insufficient explanation, or for lack of supporting rationale.

Always remember that homework is NOT meant to be an examination, it is meant to assist in your learning and development. If you need help with any HW problem, don't hesitate to ask me. You are encouraged to ask questions during my *Office Hours*, during the *TA office hours*, or through *Email to me*.

Submit Problem 5 and any three out of the remaining four problems stated below. A fifth problem may be submitted for extra credit.

1. Prove that $K_{\lfloor n/2\rfloor,\lceil n/2\rceil}$ is the only sharpness example for Mantel's Theorem. That is, show that: if G is a triangle-free n-vertex simple graph and $e(G) = \lfloor n^2/4 \rfloor$, then $G \cong K_{\lfloor n/2\rfloor,\lceil n/2\rceil}$.
Hint: Follow the proof of Mantel's Theorem. Knowing $e(G)$ allows you to conclude that some inequalities must be equalities.

2. Let $n \in \mathbb{N}$ and let d be a list of n nonnegative integers with even sum whose largest entry is less than n and differs from the smallest entry by at most 1 (e.g., 443333 or 33333322). Prove that d is graphic.
Conclude that there is a k-regular n-vertex simple graph if and only if $k < n$ and kn is even.
Comment: It might not be so easy to prove this directly, especially by induction. We "strengthened the induction hypothesis" above to include more sequences, and the resulting induction becomes much easier.

3. Let $d_1 \geq d_2 \geq \cdots \geq d_n \geq 0$. Prove that there is a loopless graph (multiple edges allowed) with degree sequence $d_1, \ldots, d_n$ if and only if $\sum d_i$ is even and $d_1 \leq \frac{1}{2}\sum d_i$.

4. Prove that every graph G has an orientation D such that $\left|d_D^+(v) - d_D^-(v)\right| \leq 1$. for every $v \in V(G)$.
Comment: Review Definition 1.4.27 in the textbook.

5. Each game of bridge involves two teams of two partners each. In a bridge club, a game can **not** be played if any two of the four people have previously been partners. At the start of the evening at this bridge club, there are 14 members present who play games until each has played exactly four times. After this, they are able to play six more games (total 12 more partnerships). Just as they are ready to wrap-up for the night, a 15th member arrives. Prove that the arrival of this new member allows at least one more game to be played.