MATH 454: Homework #8

Due Wednesday, 3/20, before 11:59pm via a single PDF file uploaded to the Homework#8 under Assignments in the Blackboard course page.

You are allowed to discuss the homework problems with no one except your classmates, the TA, and the instructor. However, the solutions should be written by you and you alone in your own words. **If you discussed HW problems with a classmate or TA, you have to write their name at the top of the HW submission as a collaborator.** Any incident of plagiarism/ cheating (from a person or from any online resource) will be strictly dealt with.

Re-read the "HW Discussion and Solution Rules" and " 'Why and How' of Homework" sections of the course information sheet for some important advice on the HWs for this course.

All problems require explicit and detailed explanations. Solutions should be written clearly, legibly, and concisely, and will be graded for both mathematical correctness and presentation. Points will be deducted for sloppiness, incoherent or insufficient explanation, or for lack of supporting rationale.

Always remember that homework is NOT meant to be an examination, it is meant to assist in your learning and development. If you need help with any HW problem, don't hesitate to ask me. You are encouraged to ask questions during my *Office Hours*, during the *TA office hours*, or through *Email to me*.

Submit any five out of the six problems stated below.

1. In a particular class, there are k students who will each complete a project. There are k projects to select from, and each student submits a list of three projects they would like to work on. You (the instructor) want the students to each take on a different project, subject to their preferences.

(a) If every project is listed by three students, prove that you can assign projects as desired.

(b) Before you have actually made the assignments, one student says that they really would like one of the projects in particular (that was on their list). Are you guaranteed to be able to assign projects to accommodate this student (while still satisfying the constraints above)? Prove that you can, or provide a situation in which you cannot.

2. For each $k \geq 2$, construct a k-regular graph with no perfect matching.

3. Two players play a game on a graph G, alternately choosing distinct vertices. Player 1 starts by choosing any vertex. Each choice of vertex in a move must be adjacent to the vertex chosen by the other player in the previous turn. That is, together the players are following a path. The last player to be able to make a move wins.

Prove that the second player has a winning strategy if G has a perfect matching, and otherwise the first player has a winning strategy.

4. Prove that every maximal matching in a graph G has size at least $\alpha'(G)/2$.

Comment: $\alpha'(G)$ denotes the size (number of edges) of maximum matching in G.

5. [An Extension of Hall's Theorem] The people, $p_1, p_2, \ldots, p_m$, in a club are planning their annual vacations. Trips $t_1, t_2, \ldots, t_n$ are available, and trip t_i has capacity c_i (total number of people allowed on that trip). Each person likes some of the trips and will travel on at most one such trip. Using this information about which people like which trips, derive a necessary and sufficient condition for being able to fill all trips to capacity with people who like them.

6. Let C be a cycle in a connected weighted graph. Let e be an edge of maximum weight on C. Prove that there is a minimum-weight spanning tree not containing e. Use this to prove that iteratively deleting a heaviest non-cut-edge until the remaining graph is acyclic produces a minimum-weight spanning tree.