MATH 554 : Homework #2

Submit five out of the following six problems.
Due Monday, February 17th.

Before starting the HW, carefully read the HW related discussion and rules described in
http://www.math.iit.edu/~kaul/TeachingSpr25/TeachingFiles/Math554.pdf

Always remember that homework is NOT meant to be an examination, it is meant to assist in your learning and development. If you are having trouble understanding a homework problem, I will be glad to direct you in the right direction without giving away the solution. You are encouraged to ask questions during the *Class*, through the *Canvas Discussion Forums*, during the *Office Hours*, or through *Email to me*.

All problems require explicit and detailed proofs/ arguments/ reasons. Solutions should be written clearly, legibly, and concisely, and will be graded for both mathematical correctness and presentation. Points will be deducted for sloppiness, incoherent or insufficient explanation, or for lack of supporting rationale.

You are allowed to discuss the homework problems with no one except your classmates, and the instructor. However, the solutions should be written by you and you alone in your own words. Any incident of plagiarism/ cheating (from a person or from any online resource) will be strictly dealt with.

If you discuss the problems with anyone, you are required to note their name at the top of your HW submission under a subtitle "Collaborator" or "Discussed with:".

1. Turán number of a graph H, the maximum size (number of edges) among all n-vertex graphs not containing H, is denoted $ex(n; H)$. Use the alteration method to prove that $ex(n; C_k) \geq \Omega(n^{1+1/(k-1)})$. In other words, prove that there exists an n-vertex graph with at least $c_k n^{1+1/(k-1)}$ edges that does not contain a k-cycle where c_k is a constant.

2. Consider Boolean formulae, $S = C_1 \wedge C_2 \wedge \ldots \wedge C_m$, that are formed by conjunction ('and' operator) of clauses where each clause, $C_i = a_1 \vee a_2 \vee \ldots \vee a_k$, is a disjunction ('or' operator) of k literals involving k distinct boolean variables. A literal is either a (Boolean) variable or its negation. Let S be a Boolean formula of the type above such that no variable occurs in more than $2^{k-\log_2 k-2}$ clauses, then prove that S is satisfiable, i.e. there exists an assignment of True/ False to the variables such that S is true.

3. The Van der Waerden number $w(l, k)$ is the least n such that every k-coloring of $[n]$ has a monochromatic l-term arithmetic progression. Use LLL to prove that $w(l, k) > (1 + o(1))(ekl)^{-1}k^l$. (Hint: You only need to give an asymptotic upper bound on the max degree of the dependency graph.) [Comment: When l is a prime, there is a construction for $w(l, 2) > l2^l$ using finite fields.]

4. Let $\{A_i\}_{i=1}^m$ and $\{B_i\}_{i=1}^m$ be subsets of [n], with $|A_i| = a_i$ and $|B_i| = b_i$, and $A_i \cap B_j = \emptyset$ if and only if $i = j$. Prove that $\sum_{i=1}^m \binom{a_i+b_i}{a_i}^{-1} \leq 1$. Apply this to prove that the maximum size of an antichain (a collection of sets s.t. no set is contained in another set) of subsets of $[n]$ is $\binom{n}{\lfloor n/2 \rfloor}$.[1] (Hint: Consider the probability space of permutations of $[n]$, equally likely, and define an appropriate event.)

5. Let G be a digraph in which every vertex has an outdegree k and indegree k. Let $r = \lfloor k/(2.25 + 2\ln k) \rfloor$. Partition $V(G)$ into r sets $V_1, \ldots, V_r$ by placing each vertex, independently, into a random V_i, chosen uniformly. Use LLL to prove that with positive probability every vertex has a successor in the set containing it. Conclude that every k-regular directed graph has a family of r pairwise disjoint cycles. (Comment: It is conjectured that $k/2$ disjoint cycles can be found, and $k/64$ is known.)

6. For $k \geq 9$, prove that every k-regular k-uniform hypergraph has a proper 2-coloring.

[1] Comment: This innocuous looking problem has deep connections with fundamental concepts and problems in classical Combinatorics. Let $\mathcal{F}$ be a collections of subsets of $[n]$ (that is, $\mathcal{F} \subseteq 2^{[n]}$, the Boolean lattice). We say $\mathcal{F}$ is an *antichain* if no set in $\mathcal{F}$ is contained in another set in $\mathcal{F}$ (in general, we can define an antichain for any collection of partially ordered sets, not just in the Boolean lattice). A natural question is to ask for the largest size of an antichain? In 1928, Sperner, of Sperner Lemma fame, in his other famous result proved that any antichain of subsets of $[n]$ has at most $\binom{n}{\lfloor n/2 \rfloor}$ sets in it, that is the middle level of the Boolean lattice is the largest antichain. This was generalized to the famous LYM inequality in 1960s: If $\mathcal{F}$ is an antichain of subsets of $[n]$ then $\sum_{A \in \mathcal{F}}(1/\binom{n}{|A|}) \leq 1$. Bollobás in 1965 generalized it to the theorem in your HW problem, which you can now prove in a few lines!

This result also has close connection to transversals, or vertex covers, of hypergraphs. We say T is a *transversal* for a hypergraph H if $T \subseteq V(H)$ and $T \cap e \neq \emptyset$ for every $e \in E(H)$, that is T 'hits' every edge in H. Like the minimum vertex cover problem for graphs, we can define the transversal number of H, $\tau(H)$, to be the minimum size of a transversal of H. H is said to be *τ-critical* if $\tau(H - e) < \tau(H)$ for every $e \in E(H)$. Then this HW problem easily implies that: the maximum size of a τ-critical r-uniform H with $\tau(H) = s + 1$ is $\binom{r+s}{s}$. Try it!