MATH 554 : Homework #4

Submit the first four problems below. Fifth problem is optional.
Due Wednesday, March 12th.

Before starting the HW, carefully read the HW related discussion and rules described in
http://www.math.iit.edu/~kaul/TeachingSpr25/TeachingFiles/Math554.pdf

Always remember that homework is NOT meant to be an examination, it is meant to assist in your learning and development. If you are having trouble understanding a homework problem, I will be glad to direct you in the right direction without giving away the solution. You are encouraged to ask questions during the *Class*, through the *Canvas Discussion Forums*, during the *Office Hours*, or through *Email to me*.

All problems require explicit and detailed proofs/ arguments/ reasons. Solutions should be written clearly, legibly, and concisely, and will be graded for both mathematical correctness and presentation. Points will be deducted for sloppiness, incoherent or insufficient explanation, or for lack of supporting rationale.

You are allowed to discuss the homework problems with no one except your classmates, and the instructor. However, the solutions should be written by you and you alone in your own words. Any incident of plagiarism/ cheating (from a person or from any online resource) will be strictly dealt with.

If you discuss the problems with anyone, you are required to note their name at the top of your HW submission under a subtitle "Collaborator" or "Discussed with:".

1. In the analysis of randomized algorithms, the following situation often occurs. A randomized algorithm computes the correct answer with probability $p > 1/2$, say $p = 2/3$. To boost the quality of the output answer, we run the algorithm n times and select the majority answer. What is the probability that this procedure is correct? Show that this procedure is correct with probability approaching 1 very rapidly.

2. Use the moment generating method to prove the following version of Chernoff-Hoeffding bounds:

   Let $X_1, \dots, X_n$ be independent random variables with $X_i \in [0,1]$ and $\mu_i = \mathbb{E}[X_i]$. Denote $S_n = \sum_{i=1}^n X_i$, $\mu = \mathbb{E}[S_n] = \sum_i \mu_i$, and $\sigma^2 = Var[S_n]$. Then $\mathbb{P}[S_n - \mu > t] \leq e^{-t^2/4\sigma^2}$ for $0 < t < 2\sigma^2$.

3. Let H be hypergraph on n vertices and m edges, then prove that the discrepancy of H, $disc(H) \leq \sqrt{2n \ln(2m)}$. What bound do you get if you apply Chebyshev's inequality? Compare the two bounds when $m = n$.

4. Let $\mathbf{a}$ be a unit vector in $\mathbb{R}^n$. Let $\mathbf{x} = \frac{1}{\sqrt{n}}(x_1, \dots, x_n) \in \mathbb{R}^n$. be chosen from the unit sphere by picking each coordinate x_i independently and uniformly at random from $\{+1, -1\}$. Let $\theta_{\mathbf{a},\mathbf{x}}$ be the angle between the vectors $\mathbf{a}$ and $\mathbf{x}$. Prove that $\mathbb{P}[|\cos(\theta_{\mathbf{a},\mathbf{x}})| > t] \leq 2e^{-nt^2/4}$.

5. [Optional Problem] There are n players in a room who play the following game. A hat is placed on the head of each player. This hat may be either red or blue. Each player sees the colors of the hats of all other players but not his/her own hat. Based on this information, each player needs to guess the color of his/her own hat.

   Let r be the number of red hats, b be the number of blue hats, $m = \max\{r, b\}$, and t be the number of players who guessed correctly (note that r, b, m, t are not known to the players).

   Design a strategy of the players that ensures t is not much smaller than m. In particular, show there is a strategy such that $t > m - O(\sqrt{n})$ with high probability.