Proposal for a course : Topics in Probabilistic Methods for Discrete Mathematics

Hemanshu Kaul
hkaul@math.uiuc.edu

May 2005

Although combinatorial problems have arisen in all fields of mathematics throughout history, discrete mathematics has only come to prominence as a mathematical discipline over the past century. It has been beneficial to adapt techniques from more mature areas of mathematics to tackle various combinatorial problems. A fruitful area of collaboration has been with probability. 'The probabilistic method' is now a well established part of the graduate curriculum in combinatorics.

Over the past two decades additional techniques with a probabilistic flavor have been developed for applications to combinatorial problems, both algorithmic and existential. A traditional first course in probabilistic combinatorics gives only the briefest hint of these areas from probability and information theory, such as Concentration of Measure, Entropy, and Rapidly Mixing Markov Chains. At UIUC, the courses on 'Methods in Combinatorics' and 'Applied Probability' do not teach these topics except for some large deviation inequalities. Moreover, the focus in Probability courses tends to be on the abstract development of the theory, which can obscure the applicability of these methods in discrete mathematics. This course will introduce graduate students to these methods and provide applications in graph theory, combinatorics, combinatorial optimization, and theoretical computer science. This course should be of interest to graduate students in Combinatorics, Probability, Operations Research, Theoretical Computer Science, and ECE. The topics are discussed in more detail below.

Concentration of Measure. Inequalities for concentration of measure are vital tools in probabilistic combinatorics, probabilistic analysis of algorithms, randomized algorithms, and stochastic combinatorial optimization. They show that the probability of a random variable being far from its mean (or median) is exponentially small, and they give bounds on probabilities of rare events. The topics include the following: introduction to concentration of measure in metric spaces and its relation to isoperimetric inequalities, Chernoff–Hoeffding bounds for sums of random variables and their generalizations, McDiarmid's method of bounded differences for Lipschitz bounded functions and its variants (leading to the Azuma–Hoeffding Martingale Inequality), and isoperimetric inequalities under Hamming metric (leading to Talagrand's convex distance isoperimetric inequality and its variants). The focus is on developing the themes underlying the various methods and illustrating the final results through applications in graph theory, combinatorial optimization and theoretical computer science. The main references, in addition to the instructor's lecture notes, include :

- N. Alon, J. Spencer, *The Probabilistic Method*, 2nd ed., (Academic Press 2000). (esp. Chapter 7)
- M. Habib, C. McDiarmid, J. Ramirez-Alfonsin, B. Reed, *Probabilistic Methods for Algorithmic Discrete Mathematics*, (Springer, 1998). (esp. McDiarmid, Concentration, 195–248)
- S. Janson, On concentration of probability, In *Contemporary Combinatorics*, ed. B. Bollobas, Bolyai Society Mathematical Studies 10 (2002), 289–301.
- C. McDiarmid, On the method of bounded differences, In *Surveys in Combinatorics*, LMS lecture note series 141 (1989), 148–188.
- J.M. Steele, *Probability theory and Combinatorial Optimization*, (SIAM, 1997).
- M. Talagrand, Concentration of measure and isoperimetric inequalities in product spaces, *Publ. Math. IHES* 81 (1995), 73–205.

Entropy. Entropy of a random variable measures the amount of uncertainty in the random variable or the amount of information obtained when the random variable is revealed. In the last decade, entropy has been applied to provide short and elegant proofs for various counting and covering problems in graphs and set systems. After introducing the elementary properties of the entropy function and Shearer's Entropy Lemma, the focus will be on combinatorial applications giving bounds on various extremal problems. Examples include Brégman's bound on the permanent of a $0,1$-matrix, bounds on the size of an intersecting family of graphs and on the number of copies of a fixed subgraph, Dedekind's problem on the number of monotone Boolean functions, and covering a complete r-uniform hypergraph with a small number of r-partite hypergraphs. We will also consider Friedgut's generalization of Shearer's Lemma, leading to a common generalization of classical inequalities such as those of Cauchy-Schwarz, Hölder, etc. The main references, in addition to the instructor's lecture notes, include :

- I. Csiszár, J. Körner, *Information Theory*, (Academic Press, 1981).
- E. Friedgut, Hypergraphs, entropy and inequalities, *The American Mathematical Monthly* 111 (2004), 749–760.
- E. Friedgut, J. Kahn, On the number of copies of one hypergraph in another, *Israel Journal of Mathematics* 105 (1998), 251–256.
- D. Galvin, P. Tetali, On weighted graph homomorphisms, *DIMACS-AMS Special Volume* 63 (2004), 97–104.
- J. Radhakrishnan, Entropy and counting, In *Computational Mathematics, Modelling and Algorithms*, ed. J.C. Mishra, (Narosa Publishers, New Delhi, 2003).
- G. Simonyi, Graph entropy - a survey, In *Combinatorial Optimization*, DIMACS Series Discrete Math. Theoret. Comput. Sci. 20 (A.M.S., 1995), 399–441.

Rapidly Mixing Markov Chains. Over the past decade the Markov chain Monte Carlo (MCMC) method has emerged as a powerful methodology for approximate counting, computing multidimensional volumes and integrals, and combinatorial optimization. The MCMC method reduces these problems to sampling over an underlying set (of solutions or combinatorial structures) w.r.t. a given distribution. This sampling is done by a Markov Chain, on the underlying set, that converges to the required (stationary) distribution. The primary step in the rigorous analysis of such an MCMC algorithm is to show that the Markov chain is rapidly mixing, i.e., it has a high rate of convergence to its stationary distribution. This analysis tends to be an interesting mix of probability and combinatorics. The topics will include the equivalence of (approximate) counting and (almost) uniform sampling, the relation between Fully Polynomial Randomized Approximation Schemes (FPRAS) and rapid mixing of Markov Chains, and the study of various methods for bounding the mixing rates of combinatorially defined Markov Chains. These methods, including coupling, conductance, and canonical paths, will be used in applications of the MCMC method to the Knapsack problem, proper colorings of a graph, linear extensions of a poset, permanent of a $0,1$-matrix, etc. The main references, in addition to the instructor's lecture notes, include :

- M. Jerrum, Mathematical foundations of the Markov chain Monte Carlo method, In *Probabilistic Methods for Algorithmic Discrete Mathematics*, (Springer, 1998), 116–165.
- M. Jerrum, *Counting, Sampling and Integrating: Algorithms and Complexity*, Lectures in Mathematics (ETH Zurich, 2003).
- M. Jerrum, A. Sinclair, The Markov chain Monte Carlo method: an approach to approximate counting and integration, In *Approximation Algorithms for NP-hard Problems*, ed. D Hochbaum, (PWS 1996), 482–520.
- L. Lovász, Random walks on graphs: a survey, In *Combinatorics, Paul Erdős is Eighty Vol. 2*, ed. D. Miklós, V. T. Sós, T. Szönyi, (Bolyai Society, 1996), 353–398.