Graph Packing and Degree Sequences

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Joint work with

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Graph Packing

Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be graphs of order at most n.

G_1 and G_2 are said to *pack* if there exist injective mappings of the vertex sets into $[n]$,
$V_i \to [n] = \{1, 2, \ldots, n\}$, $i = 1, 2$,
such that the images of the edge sets do not intersect.

- there exists a bijection $V_1 \leftrightarrow V_2$ such that $e \in E_1 \Rightarrow e \notin E_2$.
- G_1 is a subgraph of $\overline{G_2}$.

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We may assume $|V_1| = |V_2| = n$ by adding isolated vertices.

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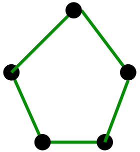
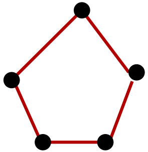
Examples and Non-Examples



C_5



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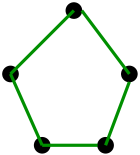

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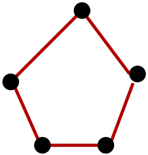

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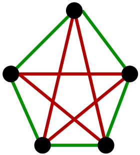

Packing

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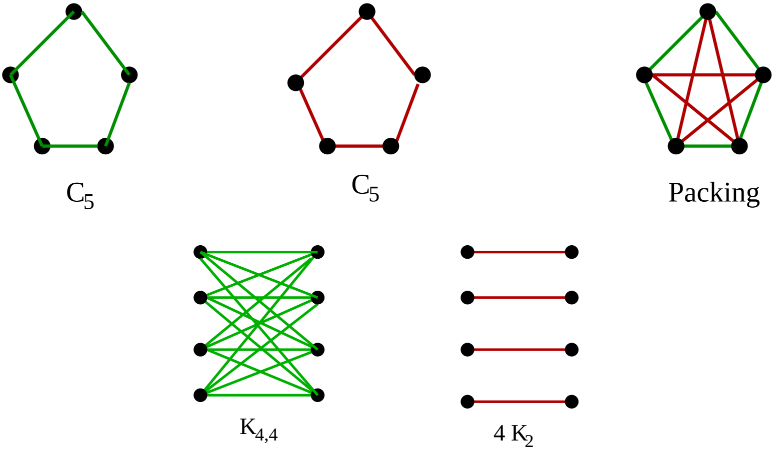

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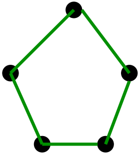

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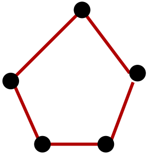

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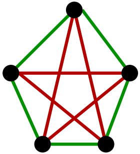

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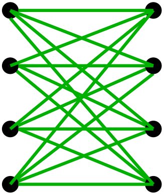
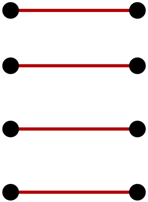
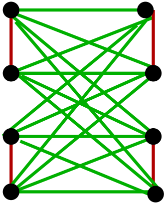


$K_{4,4}$



$4\ K_2$



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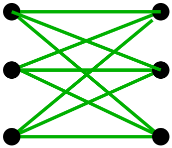
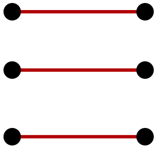


$K_{3,3}$



$3\ K_2$

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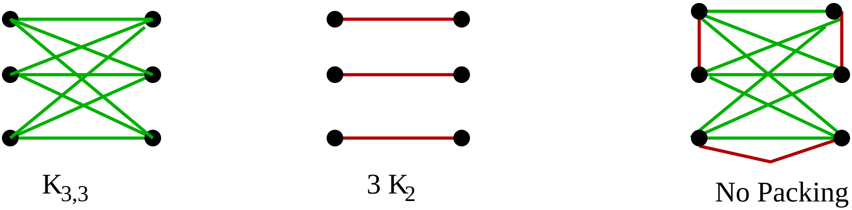

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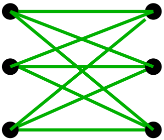

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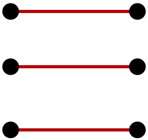

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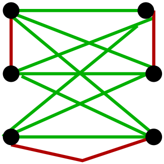
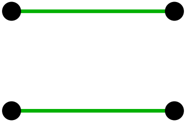


No Packing



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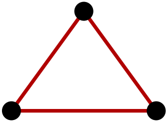

K_3

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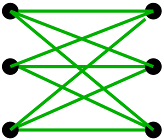

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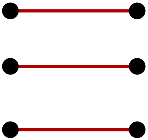

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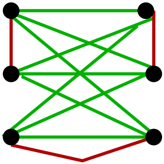

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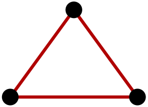

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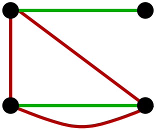

No Packing

Packing Graphs

Existence of a subgraph H in G : Whether H packs with $\overline{G}$.

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"Many" problems in Extremal Graph Theory can be interpreted as a Graph Packing problem.

- Hamiltonian Cycle in graph G : Whether the n-cycle C_n packs with $\overline{G}$.
- Equitable k-coloring of graph G : (A proper k-coloring of G such that sizes of all color classes differ by at most 1) Whether G packs with k cliques of order n/k.
- Turán-type problems : Every graph with more than $ex(n, H)$ edges must pack with $\overline{H}$.
- Ramsey-type problems.

Packing Graphs

Some examples:

Theorem: If $e(G_1)e(G_2) < \binom{n}{2}$, then G_1 and G_2 pack.

Proof. HW for Math 554 students.

Sharp for star and matching.

Theorem [Bollobas + Eldridge, 1978, & Teo + Yap, 1990]: If Δ_1, $\Delta_2 < n-1$, and $e(G_1) + e(G_2) \leq 2n-2$, then G_1 and G_2 do not pack if and only if they are one of the thirteen specified pairs of graphs.

Theorem [Sauer + Spencer, 1978] :
If $2\Delta_1\Delta_2 < n$, then G_1 and G_2 pack.

Kaul and Kostochka, 2007, characterized sharpness as: G_1 and G_2 is a perfect matching and the other either is $K_{\frac{n}{2},\frac{n}{2}}$ with $\frac{n}{2}$ odd or contains $K_{\frac{n}{2}+1}$.

Packing Graphs

Some Conjectures:

Erdős-Sos Conjecture (1963) : Let G be a graph of order n and T be a tree of size k. If $e(G) < \frac{1}{2}n(n-k)$ then T and G pack.

Known only for special classes of trees, etc.

Tree Packing Conjecture (Gyarfas $\sim$ 1968) : Any family of trees $T_2, \ldots, T_n$, where T_i has order i, can be packed.

Known for special classes of trees, and for a sequence of $n/\sqrt{2}$ such trees (Bollobas, 1983).

Bollobás-Eldridge Graph Packing Conjecture [1978] :
If $(\Delta_1 + 1)(\Delta_2 + 1) \leq n + 1$ then G_1 and G_2 pack.

Kaul, Kostochka, Yu, 2008, proved $(\Delta_1 + 1)(\Delta_2 + 1) \leq (0.6)n + 1$ suffices.

Packing Families of Graphs

Let $\mathcal{G}_1$ and $\mathcal{G}_2$ be families of graphs of order n, then $\mathcal{G}_1$ and $\mathcal{G}_2$ pack if there exists $G_1 \in \mathcal{G}_1$ and $G_2 \in \mathcal{G}_2$ such that G_1 and G_2 pack.

Note: A family $\mathcal{G}$ and its dual (the family of graphs whose complements are not in $\mathcal{G}$) cannot pack.

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The major application of graph packing results has been to proving lower bounds on computational complexity of graph properties (depth of the decision trees).

Friedgut, Kahn and Widgerson (2003) argue (and give conjectures) that results on packing of families of graphs are needed for improving such complexity bounds.

Restrictive Packing of Graph Families

Let $\mathcal{G}_1$ and $\mathcal{G}_2$ be families of labeled graphs with vertex sets all labeled as $\{v_1, \ldots, v_n\}$.

We want to find $G_1 \in \mathcal{G}_1$ and $G_2 \in \mathcal{G}_2$ such that the identity bijection between $V(G_1)$ and $V(G_2)$ gives a packing of G_1 and G_2.

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In particular, we are interested in families of graphs defined in terms of realizations of fixed degree sequences.

Degree Sequence Packing

Let $\pi_1 = (d_1^{(1)}, \ldots, d_n^{(1)})$ and $\pi_2 = (d_1^{(2)}, \ldots, d_n^{(2)})$ be graphic sequences

π_1 and π_2 pack if there exist $G_1 = G(\pi_1)$ and $G_2 = G(\pi_2)$

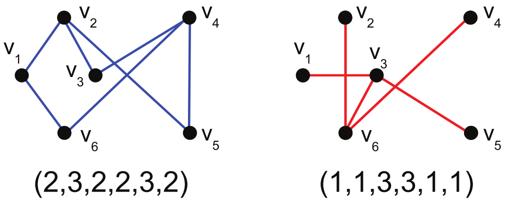
with

$$V(G_1) = V(G_2) = \{v_1, \ldots, v_n\},$$

$$E(G_1) \cap E(G_2) = \emptyset,$$

$$\deg_{G_1}(v_i) = d_i^{(1)} \text{ and } \deg_{G_2}(v_i) = d_i^{(2)}.$$

An Example



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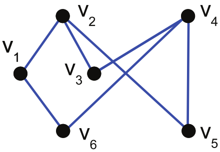
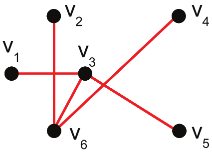
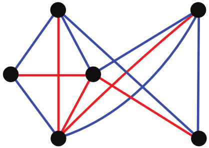


(2,3,2,2,3,2)



(1,1,3,3,1,1)



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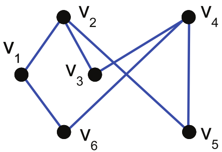

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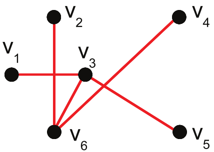

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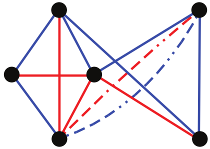

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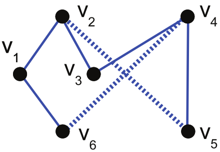

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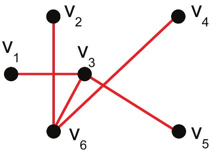

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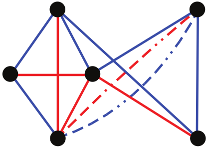

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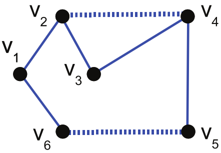

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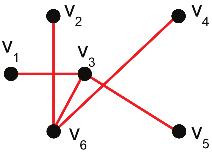

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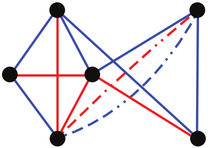

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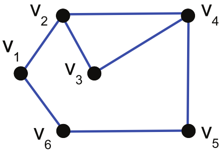

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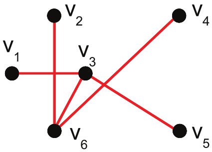

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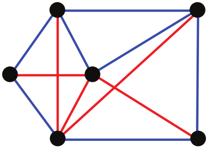

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Applications

Feasibility of supply-demand requirements of multiple goods (a restrictive form of multi-commodity flow) in a (bipartite) network is equivalent to packing the corresponding (bipartite) degree sequences.

Applications

Discrete tomography is concerned with reconstructing discrete objects, such as the atomic structure of a crystalline lattice.

In 2-dimensions, we assume that our crystalline structure lies on the integer lattice, and X-rays are projected on the horizontal and the vertical.

The X-rays return information on the number of atoms of specific types in each row/column. Then a feasible solution to this reconstruction problem is equivalent to packing two bipartite degree sequences.

Applications

Theorem (Dürr, Guiñez, Matamala, 2009)

The discrete tomography reconstruction problem is NP-Complete for $c \geq 2$ atom types.

A straightforward reduction demonstrates that determining if 2 or more graphic sequences pack is also NP-complete.

A Graph Packing Result

Theorem (Sauer–Spencer, 1978)

Let G_1 and G_2 be n-vertex graphs with max degrees Δ_1 and Δ_2.

If $\Delta_1\Delta_2 < n/2$, then G_1 and G_2 pack.

A Graph Packing Result

Theorem (Busch, Ferrara, Hartke, Jacobson, Kaul, West, 2011)

Let $\pi_1 = (d_1^{(1)}, \ldots, d_n^{(1)})$ and $\pi_2 = (d_1^{(2)}, \ldots, d_n^{(2)})$ be graphic sequences.

If $\Delta = \max\{d_i^{(1)} + d_i^{(2)}\}$ and $\delta = \min\{d_i^{(1)} + d_i^{(2)}\}$
are such that $\Delta \leq \sqrt{2\delta n} - (\delta - 1)$,

then π_1 and π_2 pack, except that strict inequality is required when $\delta = 1$.

This result is sharp for all δ.

Comparing the Results

Sauer–Spencer : $\Delta_1\Delta_2 < n/2 \Rightarrow G_1$ and G_2 pack.

BFHJKW : (with $\delta = 1$)

$$\max\{d_i^{(1)} + d_i^{(2)}\} < \sqrt{2n} \Rightarrow \pi_1 \text{ and } \pi_2 \text{ pack.}$$

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$$\Delta_1 + \Delta_2 < \sqrt{2n} \Rightarrow \Delta_1\Delta_2 < n/2$$

A Direct Analogue to Sauer-Spencer

We conjecture the following, which would be a more direct analogue to the Sauer-Spencer Theorem.

Conjecture

Let $\pi_1 = (d_1^{(1)}, \dots, d_n^{(1)})$ and $\pi_2 = (d_1^{(2)}, \dots, d_n^{(2)})$ be graphic sequences with δ the least entry in $\pi_1 + \pi_2$.

If $\delta \geq 1$ and $\max\{d_i^{(1)} d_i^{(2)}\} < n/2$, then π_1 and π_2 pack.

Kundu's k-Factor Theorem

When necessary conditions are sufficient for packing:

Kundu, 1973

Let k be a positive integer, and let π_1 and π_2 be graphic sequences such that each term in π_2 is k.

Then π_1 and π_2 pack if and only if $\pi_1 + \pi_2$ is graphic.

Alternatively, if $\pi = (d_1, \ldots, d_n)$ is a graphic sequence such that $\pi - k = (d_1 - k, \ldots, d_n - k)$ is graphic, then there exists a realization G of π that has a k-factor.

Recall, k-factor is a k-regular spanning subgraph.

Kundu's k-Factor Theorem

When necessary conditions are sufficient for packing:

Kundu, 1973

Let k be a positive integer, and let π_1 and π_2 be graphic sequences such that each term in π_2 is k.

Then π_1 and π_2 pack if and only if $\pi_1 + \pi_2$ is graphic.

Alternatively, if $\pi = (d_1, \ldots, d_n)$ is a graphic sequence such that $\pi - k = (d_1 - k, \ldots, d_n - k)$ is graphic,

then there exists a realization G of π that has a k-factor.

Recall, k-factor is a k-regular spanning subgraph.

Extending Kundu's Theorem

Rao and Rao showed the following while attempting to prove the (then) k-factor conjecture.

A.R. Rao and S.B. Rao, 1972

Let $\pi = (d_1, \ldots, d_n)$ be a graphic sequence such that $\pi - k = (d_1 - k, \ldots, d_n - k)$ is graphic for some $k > 0$.

Then for any nonnegative integer $r \leq k$ such that rn is even, $\pi - r = (d_1 - r, \ldots, d_n - r)$ is also graphic.

Therefore, if some realization of π has a k-factor, then there is also a realization that contains an r-factor for any (feasible) $r < k$.

A Conjecture

We conjecture that Kundu's Theorem can be strengthened in the following manner.

Conjecture

Let $k > 0$ and let $\pi = (d_1, \ldots, d_n)$ be a graphic sequence such that $\pi - k = (d_1 - k, \ldots, d_n - k)$ is graphic.

Then for any $k_1, \ldots, k_t$ such that nk_i is even for all i and

$$k_1 + k_2 + \ldots k_t = k,$$

there is a realization G of π containing edge-disjoint subgraphs $F_1, \ldots, F_t$ such that each F_i is a k_i-factor of G.

In other words, there is a realization G of π containing a k-factor that can be decomposed into k_i-factors.

Odd Order

If n is odd, then each k_i (and hence k) must be even.

Since any $2r$-regular graph has a 2-factorization, the conjecture for n odd follows from Kundu's Theorem.

It would therefore be sufficient to prove the following:

Conjecture

Let n be even and let $\pi = (d_1, \ldots, d_n)$ be a graphic seq such that $\pi - k = (d_1 - k, \ldots, d_n - k)$ is graphic for some $k > 0$.

Then there exists a realization G of π that contains k edge-disjoint 1-factors.

We recently learnt that this was originally conjectured by R. Brualdi in 1976. No progress has been reported so far.

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Bounded Maximum Degree

It is straightforward to verify the conjecture when the largest term in π is bounded.

Theorem (Busch, Ferrara, Hartke, Jacobson, Kaul, West, 2011)

Let n be even and let $\pi = (d_1, \ldots, d_n)$ be a graphic sequence such that $\pi - k = (d_1 - k, \ldots, d_n - k)$ is graphic for some $k > 0$

$$\text{and } \max d_i \leq \frac{n}{2} + 1.$$

Then there exists a realization G of π that contains k edge-disjoint 1-factors.

Proof

The proof is by induction on k, with $k = 1$ following from Kundu's Theorem.

Suppose that $\pi = (d_1, \ldots, d_n)$ and $\pi - k = (d_1 - k, \ldots, d_n - k)$ are both graphic, $k \geq 2$.

By Rao and Rao, $\pi - 2 = (d_1 - 2, \ldots, d_n - 2)$ is graphic.

By induction, there is a realization $G = G(\pi - 2)$ with $k - 2$ edge-disjoint 1-factors.

$$\Delta(G) \leq \frac{n}{2} - 1 \Longrightarrow \delta(\overline{G}) \geq \frac{n}{2} \Longrightarrow \overline{G} \text{ is hamiltonian.}$$

Decompose hamiltonian cycle in $\overline{G}$ into two 1-factors; add these to G.

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Large Minimum Degree

The previous theorem applied to the complementary degree sequence gives:

Corollary (Busch, Ferrara, Hartke, Jacobson, Kaul, West, 2011)

Fix k, n with n even, and $\pi = (d_1, \ldots, d_n)$ be a graphic sequence such that $\pi - k = (d_1 - k, \ldots, d_n - k)$ is graphic for some $k > 0$

If every entry in π is at least $n/2 + k - 2$, then some realization of π has k edge-disjoint 1-factors.

(k-2)-factor and 1-factors

Theorem (Busch, Ferrara, Hartke, Jacobson, Kaul, West, 2011)

Let n be even and let $\pi = (d_1, \ldots, d_n)$ be a graphic sequence such that $\pi - k = (d_1 - k, \ldots, d_n - k)$ is graphic for some $k > 0$

Then there exists a realization of π containing 1-factors F_1 and F_2, and a $(k-2)$-factor F_{k-2} that are edge-disjoint.

The proof utilizes the Gallai-Edmonds decomposition and edge exchanges (2-switches), plus some new ideas.

Conjecture for $k \leq 3$

As a consequence we get that the conjecture is true for $k \leq 3$

Theorem (Busch, Ferrara, Hartke, Jacobson, Kaul, West, 2011)

Let n be even and let $\pi = (d_1, \ldots, d_n)$ be a graphic sequence such that $\pi - 3 = (d_1 - 3, \ldots, d_n - 3)$ is graphic

Then there exists a realization of π containing three edge-disjoint 1-factors.

An Illustration of the Proof

Let me illustrate a part of the proof for $k = 3$.

Assume that we have already found a realization of π with a 3-factor that can be decomposed into a 2-factor $\mathcal{F}$ and 1-factor (this itself takes some work using Gallai-Edmonds Decomposition and edge exchanges).

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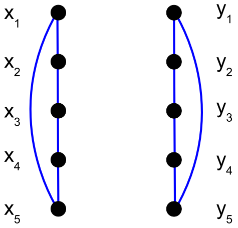
If every component of $\mathcal{F}$ is of even order then $\mathcal{F}$ decomposes into two edge-disjoint 1-factors, and the result follows.

Suppose not, since n is even, $\mathcal{F}$ must have at least two odd components, two odd cycles.

An Illustration of the Proof



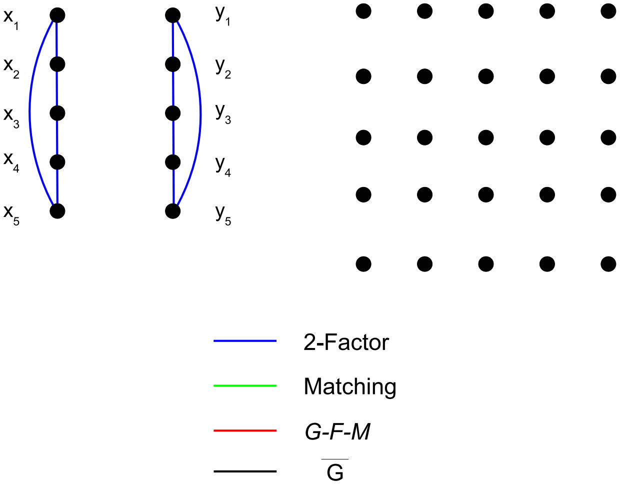
2-Factor

Matching

G-F-M

$\overline{G}$

An Illustration of the Proof



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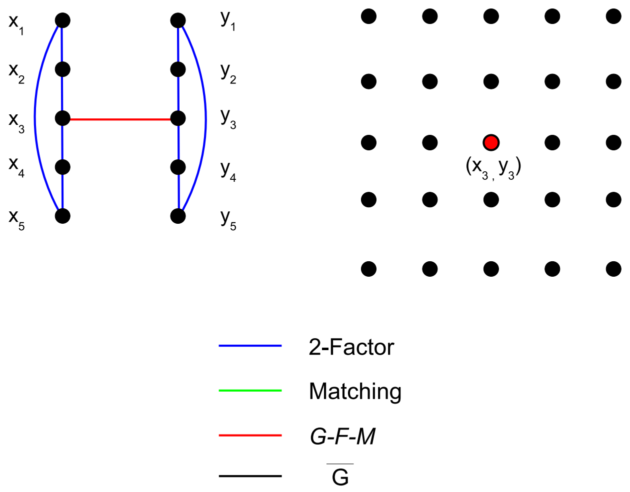

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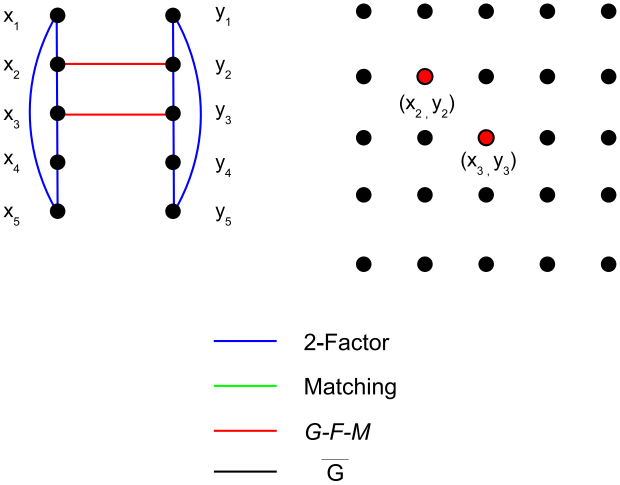

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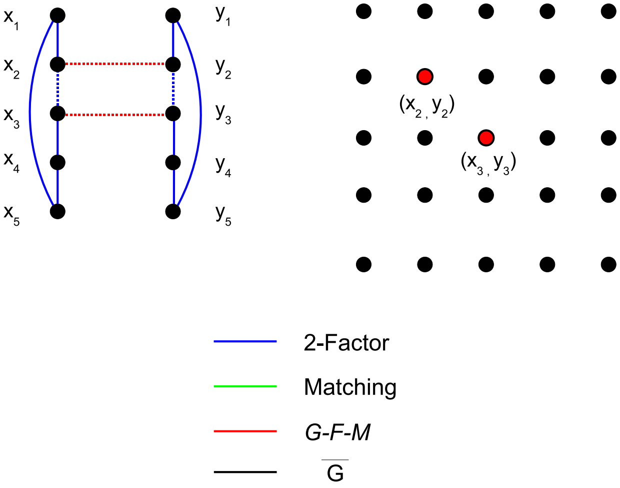

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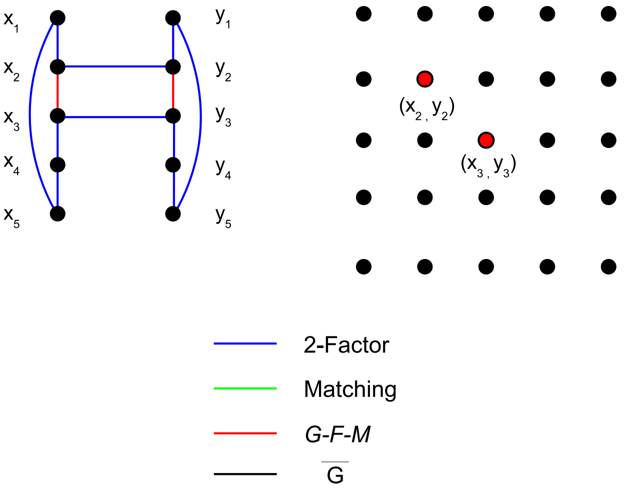

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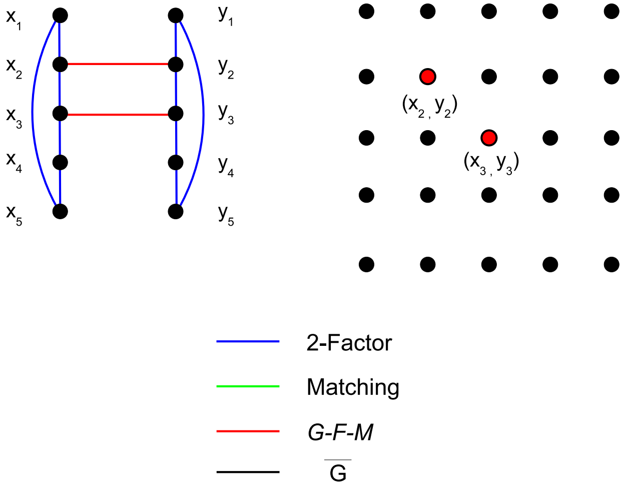

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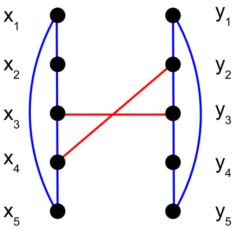

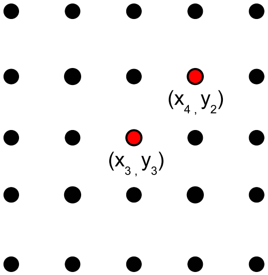

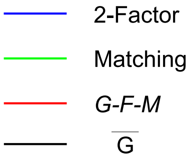

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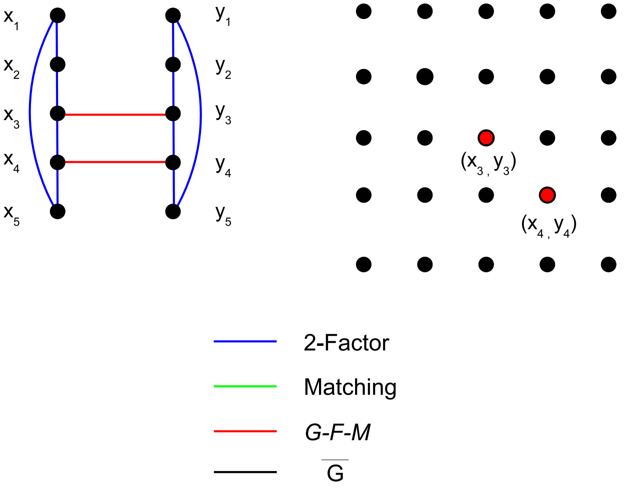

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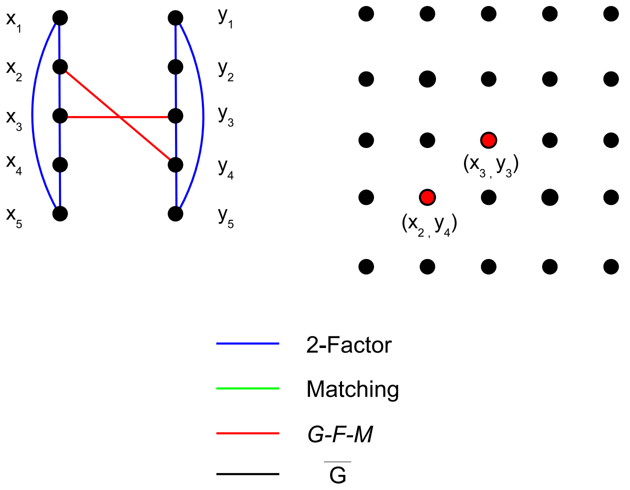

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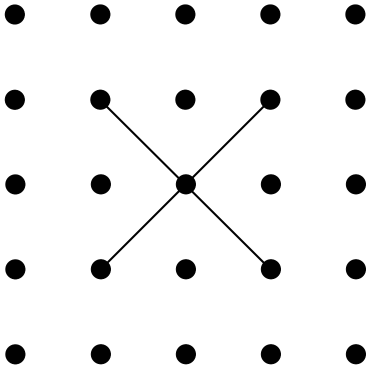

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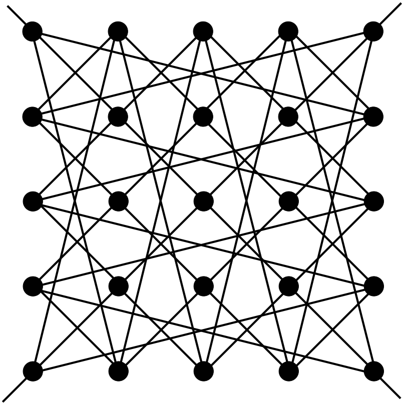

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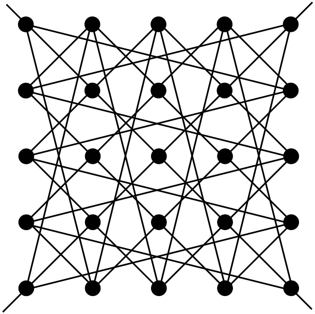
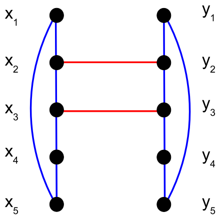
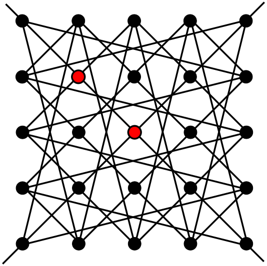


Aim: Give a proper coloring of $Aux(G)$ with green (matching $\mathcal{M}$) and red ($G - \mathcal{F} - \mathcal{M}$) and black ($\overline{G}$).

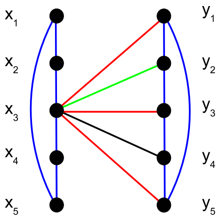
An Illustration of the Proof

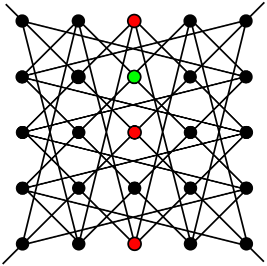




If we cannot give this coloring then a forbidden edge-swap must exist.

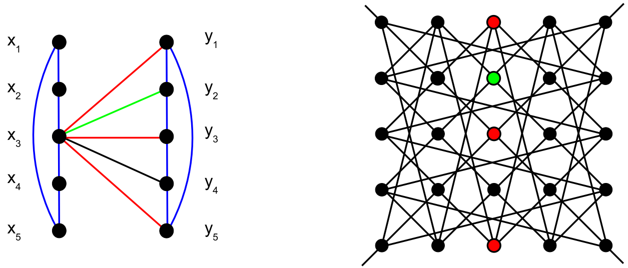
An Illustration of the Proof





Each row (and column) in $Aux(G)$ corresponds to a vertex in one of the odd cycles.

An Illustration of the Proof



Hence color green (corresponding to the matching $\mathcal{M}$) can be used at most once in each row and column.

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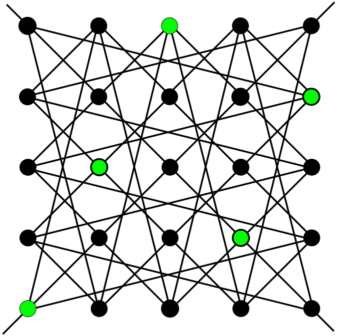

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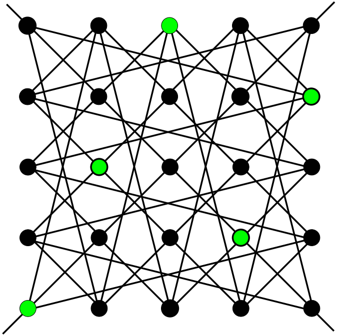
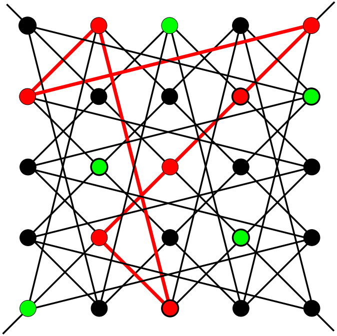


The proof concludes by showing that if at most one vertex in each row and column of $Aux(G)$ is colored green, and no two of these green vertices are adjacent, then there is an odd cycle amongst the other vertices.

An Illustration of the Proof



This allows us to do a edge-swap in G and combine the two odd cycles into an even cycle.