Distinguishing Chromatic Number of Graphs

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In $K_n$, it is impossible to distinguish between any two vertices, $u$ and $v$, because they are structurally identical.

More formally, $\text{Aut}(K_n) = S_n$, the Symmetric group formed by all the permutations on $n$ objects.

Similarly, $\text{Aut}(C_n) = D_{2n}$, the Dihedral group formed by rotations and flips.
If we want to distinguish vertices in $K_n$, we have to give each of them a distinct name (label). So, $K_n$ needs $n$ labels.

But, many times we can get away with using far less number of labels.
“Suppose you have a key ring with $n$ identical looking keys. You wish to label the handles of the keys in order to tell them apart. How many labels will you need?”

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We want to figure out how many labels we need to distinguish vertices in $C_n$.

$C_3$, $C_4$, and $C_5$ need three labels.

When $n \geq 6$, $C_n$ needs only two labels!!

So we want to be able to ‘decode’ the ‘real identity’ of a vertex using only these (few) labels and the structure of the graph.
A distinguishing $k$-labeling of $G$ is a labeling of $V(G)$ with $k$ labels such that the only color-preserving automorphism of $G$ is the identity.

**Distinguishing Number**, $D(G)$, is the least $k$ such that $G$ has a distinguishing $k$-labeling.

Introduced by Albertson and Collins in 1996.

Since then, a whole class of research literature combining graphs and group actions has arisen around this topic.
Some examples:

\[ D(G) = 1 \text{ if and only if } Aut(G) = \{\text{identity}\} \]

\[ D(K_n) = D(K_{1,n}) = n. \text{ Both have } Aut(G) = S_n. \]

It is possible to construct a graph \( G \) with \( Aut(G) = S_n \) and \( D(G) = \sqrt{n} \).

\[ D(K_{n,n}) = n + 1. \]

\( D(C_n) \) equals 3 if \( 3 \leq n \leq 5 \), and equals 2 if \( n \geq 6 \).

\[ D(P_n) = 2. \]
In general the value of the Distinguishing number is strongly influenced by the relevant Automorphism group, rather than the particular graph.

For a group $\Gamma$, $D(\Gamma) = \{D(G) : \text{Aut}(G) \cong \Gamma, G \text{ graph}\}$

**Theorem** [Albertson + Collins, 1996]
$D(D_{2n}) = \{2\}$ unless $n = 3, 4, 5, 6, 10$, in which case, $D(D_{2n}) = \{2, 3\}$.

**Theorem** [Tymoczko, 2004]
$D(S_n) \subseteq \{2, 3, \ldots, n\}$.

**Conjecture** [Klavzar+ Wong + Zhu, 2005]
$D(S_n) = \{\lfloor n^{1/k} \rfloor : k \in \mathbb{Z}^+\}$. 
Distinguishing numbers tend to be fixed numbers that depend more on the automorphism structure than the graph structure.

We want a proper coloring (not just an unrestricted labeling) that breaks all the symmetries of a graph, identifying each of its vertices uniquely.
For a university semester, we could define a ‘conflict’ graph on courses, where each course is a vertex, and edges occur between pairs of vertices corresponding to courses with overlapping time.

We want to assign rooms to courses (through a proper coloring). .... and more
Distinguishing through proper colorings

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We want to assign rooms to courses (through a proper coloring). .... and more

“Find a coloring of the conflict graph that uniquely identifies each course as well as specifying the room each would use.”

We not only ‘decode’ the ‘real identity’ of a vertex using only these (few) labels and the structure of the graph, but get a useful partition of the vertices into ‘conflict-free’ subsets.
A distinguishing proper $k$-coloring of $G$ is a proper $k$-coloring of $G$ such that the only color-preserving automorphism of $G$ is the identity.

Distinguishing Chromatic Number, $\chi_D(G)$, is the least $k$ such that $G$ has a distinguishing proper $k$-coloring.

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**Theorem** [Collins + Trenk, 2006]

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\chi_D(G) \leq 2\Delta(G), \text{ with equality iff } G = K_{\Delta,\Delta} \text{ or } C_6.
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Note that the chromatic number, $\chi(G)$, is an immediate lower bound for $\chi_D(G)$. 


Not Distinguishing
Examples

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Distinguishing

\[\chi_D(P_{2n+1}) = 3 \quad \text{and} \quad \chi_D(P_{2n}) = 2\]
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Examples

\[ \chi_D(P_{2n+1}) = 3 \text{ and } \chi_D(P_{2n}) = 2 \]

\[ \chi_D(C_n) = 3 \text{ except } \chi_D(C_4) = \chi_D(C_6) = 4 \]
Motivating Question

When are $D(G)$ and $\chi_D(G)$ small?
Just one more than the minimum allowed?
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When are $D(G)$ and $\chi_D(G)$ small?
Just one more than the minimum allowed?

Find a large general class of graphs for which

$$D(G) \leq 1 + 1$$

$$\chi_D(G) \leq \chi(G) + 1$$

Our answer will be in terms of Cartesian Product of Graphs.
A graph $G$ is said to be a **prime graph** if whenever $G = G_1 \square G_2$, then either $G_1$ or $G_2$ is a singleton vertex.

**Prime Decomposition Theorem** [Sabidussi(1960) and Vizing(1963)] Let $G$ be a connected graph, then $G \cong G_1^{p_1} \square G_2^{p_2} \square \ldots \square G_d^{p_d}$, where $G_i$ and $G_j$ are distinct prime graphs for $i \neq j$, and $p_i$ are constants.

**Theorem** [Imrich(1969) and Miller(1970)] All automorphisms of a cartesian product of graphs are induced by the automorphisms of the factors and by transpositions of isomorphic factors.
Fact: Let $G = \square_{i=1}^{d} G_i$. Then $\chi(G) = \max_{i=1,\ldots,d} \{\chi(G_i)\}$

Let $f_i$ be an optimal proper coloring of $G_i$, $i = 1, \ldots, d$.

Canonical Coloring $f^d : V(G) \to \{0, 1, \ldots, t - 1\}$ as

$$f^d(v_1, v_2, \ldots, v_d) = \sum_{i=1}^{d} f_i(v_i) \mod t , \quad t = \max_{i} \{\chi(G_i)\}$$

Notation: $G^d = \square_{i=1}^{d} G$
Theorem [Bogstad + Cowen, 2004]
\[ D(Q_d) = 2, \text{ for } d \geq 4, \text{ and } D(Q_2) = D(Q_3) = 3 \]
where \( Q_d \) is the \( d \)-dimensional hypercube.

Theorem [Albertson, 2005]
\[ D(G^d) = 2, \text{ for } d \geq 4, \text{ if } G \text{ is a prime graph.} \]

Theorem [Klavzar + Zhu, 2006]
\[ D(G^d) = 2, \text{ for } d \geq 3. \]

Follows from \( D(K_n^d) = 2 \), proved using a probabilistic argument (when automorphisms of \( G \) have few fixed points then \( D(G) \) is large).
Recall, $\chi(G) \leq \chi_D(G)$

In general, $\chi_D(G)$ might need many more colors than $\chi(G)$.

**Theorem** [Collins + Trenk, 2006]
$\chi_D(G) = n(G) \iff G$ is a complete multipartite graph.

$\chi_D(K_{n_1,n_2,\ldots,n_t}) = \sum_{i=1}^{t} n_i$ while $\chi(K_{n_1,n_2,\ldots,n_t}) = t$,

Making our task more difficult.
Theorem [Choi + Hartke + K., 2006+]
Given $t_i \geq 2$, $\chi_D(\Box_{i=1}^{d} K_{t_i}) \leq \max_i \{t_i\} + 1$, for $d \geq 5$. 
Theorem [Choi + Hartke + K., 2006+]
Given $t_i \geq 2$, $\chi_D(\square_{i=1}^{d} K_{t_i}) \leq \max_i \{t_i\} + 1$, for $d \geq 5$.

Corollary: Given $t \geq 2$, $\chi_D(K_t^d) \leq t + 1$, for $d \geq 5$.

Both these upper bounds are 1 more than their respective lower bounds.
Hamming Graphs and Hypercubes

Theorem [Choi + Hartke + K., 2006+]
Given $t_i \geq 2$, \( \chi_D(\square^d_{i=1} K_{t_i}) \leq \max_i \{t_i\} + 1 \), for $d \geq 5$.

Corollary: Given $t \geq 2$, \( \chi_D(K^d_t) \leq t + 1 \), for $d \geq 5$.

Both these upper bounds are 1 more than their respective lower bounds.

These results allow us to exactly determine the distinguishing chromatic number of hypercubes.

Corollary: \( \chi_D(Q_d) = 3 \), for $d \geq 5$. 
Main Theorem

Theorem [Choi + Hartke + K., 2006+]
Let $G$ be a graph. Then there exists an integer $d_G$ such that for all $d \geq d_G$, $\chi_D(G^d) \leq \chi(G) + 1$.

By the Prime Decomposition Theorem for Graphs, $G = G_1^{p_1} \square G_2^{p_2} \square \ldots \square G_k^{p_k}$, where $G_i$ are distinct prime graphs. (This prime decomposition can be found in polynomial time)

Then, $d_G = \max_{i=1,\ldots,k} \left\{ \frac{\lg n(G_i)}{p_i} \right\} + 5$

Note, $n(G) = (n(G_1))^{p_1} \cdot (n(G_2))^{p_2} \cdots \cdot (n(G_k))^{p_k}$

At worst, $d_G = \lg n(G) + 5$ suffices.
Main Theorem

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Let $G$ be a graph. Then there exists an integer $d_G$ such that for all $d \geq d_G$, $\chi_D(G^d) \leq \chi(G) + 1$.

\[ d_G = \max_{i=1,\ldots,k} \left\{ \frac{\lg n(G_i)}{p_i} \right\} + 5 \]

when, $n(G) = (n(G_1))^{p_1} \ast (n(G_2))^{p_2} \ast \cdots \ast (n(G_k))^{p_k}$

$d_G$ is unlikely to be a constant, as the example of Complete Multipartite Graphs indicates –

pushing $\chi_D(K_{n_1,n_2,\ldots,n_t})$ down from $n(G)$ to $t + 1$ can not happen with only a fixed number of products.
Proof Idea for Main Theorem

Fix an optimal proper coloring of $G$.

Embed $G$ in a complete multipartite graph $H$.

Form $H$ by adding all the missing edges between the color classes of $G$.

Now work with $H$.

BUT $G \subseteq H \not\Rightarrow \chi_D(G) \leq \chi_D(H)$!
Fix an optimal proper coloring of $G$.

Embed $G$ in a complete multipartite graph $H$.

Form $H$ by adding all the missing edges between the color classes of $G$.

Then construct a distinguishing proper coloring of $H^d$ that is also a distinguishing proper coloring of $G^d$.

We identify (relatively small) subgraphs which, when distinguished, can be used to distinguish the parent graph.
Conjecture: For any fixed $\rho > 0$, there exists a graph $G$ such that $\chi_D(G^d) > \chi(G) + 1$ for $d \leq \rho$. 
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Problem: Characterize graphs $G$ with $\chi_D(G^d) = \chi(G)$. 
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Problem: Characterize graphs $G$ with $\chi_D(G^d) = \chi(G)$.

Problem: Find other families of graphs for which the distinguishing chromatic number is close to its chromatic number.

Theorem [Collins + Hovey + Trenk, 2008+] If $Aut(G) = \mathbb{Z}_p^n$ then $\chi_D(G) \leq \chi(G) + 1$. 