Graph Packing and a Generalization of the Theorems of Sauer-Spencer and Brandt

Hemanshu Kaul

Illinois Institute of Technology

www.math.iit.edu/∼kaul

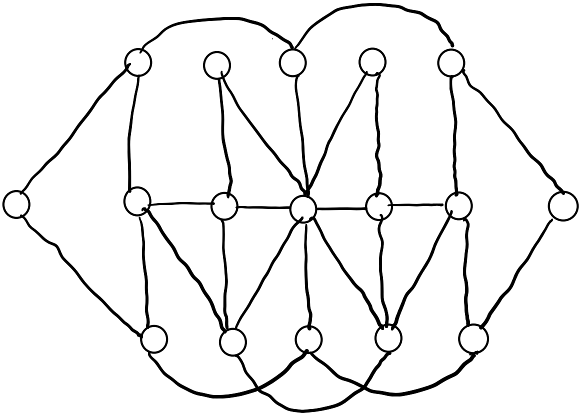
kaul@iit.edu

Joint work with

Benjamin Reiniger

A Puzzle

Can you fill in the numbers $1, 2, \ldots, 17$ in the 17 circles below without repetition so that no two consecutive numbers are placed in circles with a line segment joining them?



Graph Packing

- $G_1 = (V_1, E_1)$ and $G_2 = (V_1, E_1)$, two n-vertex graphs are said to pack if there exist injective mappings of the vertex sets into $[n]$,
  $V_i \to [n] = \{1, 2, \ldots, n\}$, $i = 1, 2$,
  such that the images of the edge sets do not intersect.
- Equivalently, there exists a bijection $V_1 \leftrightarrow V_2$ such that $e \in E_1 \Rightarrow e \notin E_2$.
- G_1 is a subgraph of $\overline{G_2}$, the complement of G_2.
- This definition is easily generalizable to more than two graphs, or to hypergraphs, etc.

Graph Packing

- $G_1 = (V_1, E_1)$ and $G_2 = (V_1, E_1)$, two n-vertex graphs are said to pack if there exist injective mappings of the vertex sets into $[n]$,
  $V_i \to [n] = \{1, 2, \ldots, n\}$, $i = 1, 2$,
  such that the images of the edge sets do not intersect.
- Equivalently, there exists a bijection $V_1 \leftrightarrow V_2$ such that $e \in E_1 \Rightarrow e \notin E_2$.
- G_1 is a subgraph of $\overline{G_2}$, the complement of G_2.
- This definition is easily generalizable to more than two graphs, or to hypergraphs, etc.

Graph Packing

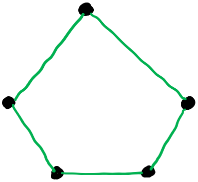
- $G_1 = (V_1, E_1)$ and $G_2 = (V_1, E_1)$, two n-vertex graphs are said to pack if there exist injective mappings of the vertex sets into $[n]$,
  $V_i \to [n] = \{1, 2, \ldots, n\}$, $i = 1, 2$,
  such that the images of the edge sets do not intersect.
- Equivalently, there exists a bijection $V_1 \leftrightarrow V_2$ such that $e \in E_1 \Rightarrow e \notin E_2$.
- G_1 is a subgraph of $\overline{G_2}$, the complement of G_2.
- This definition is easily generalizable to more than two graphs, or to hypergraphs, etc.

Graph Packing

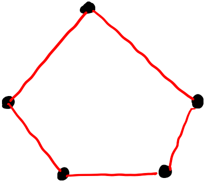
- $G_1 = (V_1, E_1)$ and $G_2 = (V_1, E_1)$, two n-vertex graphs are said to pack if there exist injective mappings of the vertex sets into $[n]$,
  $V_i \to [n] = \{1, 2, \ldots, n\}$, $i = 1, 2$,
  such that the images of the edge sets do not intersect.
- Equivalently, there exists a bijection $V_1 \leftrightarrow V_2$ such that $e \in E_1 \Rightarrow e \notin E_2$.
- G_1 is a subgraph of $\overline{G_2}$, the complement of G_2.

- This definition is easily generalizable to more than two graphs, or to hypergraphs, etc.

Examples & Non-Examples

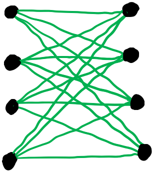


C_5

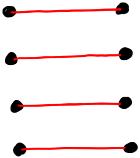


C_5

Packing?



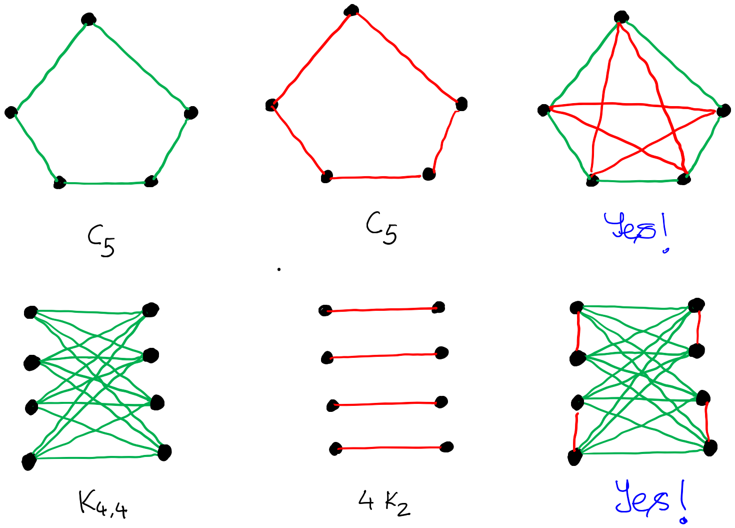
$K_{4,4}$



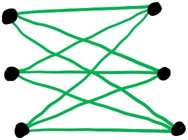
$4K_2$

Packing?

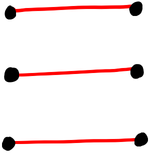
Examples & Non-Examples



Examples & Non-Examples

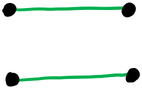


$K_{3,3}$

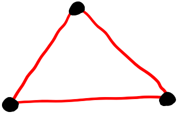


$3K_2$

Packing?



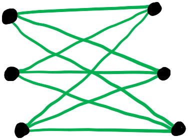
$2K_2$



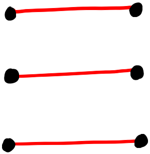
K_3

Packing?

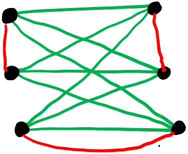
Examples & Non-Examples



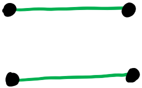
$K_{3,3}$



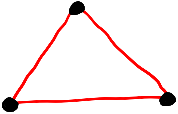
$3K_2$



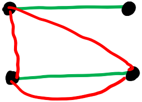
No!



$2K_2$



K_3



No!

A Common Generalization

- Hamiltonian Cycle in graph G : Whether the n-cycle C_n packs with $\overline{G}$.
- The independence number $\alpha(G)$ of an n-vertex graph G is at least k if and only if G packs with $K_k + K_{n-k}$.
- Proper k-coloring of n-vertex graph G : Whether G packs with an n-vertex graph that is the union of k cliques.
- Equitable k-coloring of n-vertex graph G : Whether G packs with complement of the Turán Graph $T(n, k)$.
- Turán-type problems : Every graph with more than $ex(n, H)$ edges must pack with $\overline{H}$.
- Ramsey-type problems.
- "most" problems in Graph Theory.

A Distinction

- In packing problems, each member of a ‘large’ family of graphs contains each member of another ‘large’ family of graphs.

A Distinction

- In packing problems, each member of a ‘large’ family of graphs contains each member of another ‘large’ family of graphs.
- In subgraph problems, (usually) at least one of the two graphs is fixed.

A Distinction

- In packing problems, each member of a 'large' family of graphs contains each member of another 'large' family of graphs.

Theorem

If $e(G_1)e(G_2) < \binom{n}{2}$, then G_1 and G_2 pack.

A Distinction

- In packing problems, each member of a ‘large’ family of graphs contains each member of another ‘large’ family of graphs.

Theorem

If $e(G_1)e(G_2) < \binom{n}{2}$, then G_1 and G_2 pack.

Proof. Pick a random bijection between $V(G_1)$ and $V(G_2)$, uniformly among the set of all $n!$ such bijections.

Sharp for $G_1 = S_{2m}$, star of order $2m$, and $G_2 = mK_2$, matching of size m, where $n = 2m$.

A Distinction

- In packing problems, each member of a 'large' family of graphs contains each member of another 'large' family of graphs.

Theorem (Bollobás, Eldridge (1978), & Teo, Yap (1990))

If $\Delta_1, \Delta_2 < n-1$, and $e(G_1) + e(G_2) \leq 2n-2$, then G_1 and G_2 do not pack if and only if they are one of the thirteen specified pairs of graphs.

A Distinction

- In packing problems, each member of a 'large' family of graphs contains each member of another 'large' family of graphs.

Conjecture (Erdős-Sós (1962))

Let G be a graph of order n and T be a tree of size k. If $e(G) < \frac{1}{2}n(n-k)$ then T and G pack.

A Distinction

- In packing problems, each member of a 'large' family of graphs contains each member of another 'large' family of graphs.

Conjecture (Erdős-Sós (1962))

Let G be a graph of order n and T be a tree of size k.
If $e(G) < \frac{1}{2}n(n-k)$ then T and G pack.

- Each graph with more than $\frac{1}{2}n(k-1)$ edges contains every tree of size k.
  This says average degree k guarantees every tree of size k.
  The corresponding minimum degree result is easy.

A Distinction

- In packing problems, each member of a ‘large’ family of graphs contains each member of another ‘large’ family of graphs.

Conjecture (Erdős-Sós (1962))

Let G be a graph of order n and T be a tree of size k. If $e(G) < \frac{1}{2}n(n-k)$ then T and G pack.

- Sharp, if true. Take disjoint copies of k-cliques.
  Known only for special classes of trees, etc.

Bollobás-Eldridge-Catlin Conjecture

Conjecture (Bollobás, Eldridge (1978), & Catlin (1976))

If $(\Delta_1 + 1)(\Delta_2 + 1) \leq n + 1$ *then* G_1 *and* G_2 *pack.*

- If $\delta(G) > \frac{kn-1}{k+1}$, then
  G contains all graphs with maximum degree at most k.
- If true, this conjecture would be sharp:
  $\Delta_2 K_{\Delta_1+1} + K_{\Delta_1-1}$ and $\Delta_1 K_{\Delta_2+1} + K_{\Delta_2-1}$

Bollobás-Eldridge-Catlin Conjecture

Conjecture (Bollobás, Eldridge (1978), & Catlin (1976))
If $(\Delta_1 + 1)(\Delta_2 + 1) \leq n + 1$ then G_1 and G_2 pack.

- If true, this conjecture would be a considerable extension of

Theorem (Hajnal-Szemerédi (1971))
Every graph G has an equitable k-coloring for $k \geq \Delta(G) + 1$.

Equitable colorings of graphs have been used to

- extend Chernoff-Hoeffding concentration bounds to dependent random variables (Pemmaraju, 2003)
- extend Arnold-Groeneveld order statistics bounds to dependent random variables (Kaul, Jacobson, 2006)

Bollobás-Eldridge-Catlin Conjecture

Conjecture (Bollobás, Eldridge (1978), & Catlin (1976))
If $(\Delta_1 + 1)(\Delta_2 + 1) \leq n + 1$ then G_1 and G_2 pack.

- The conjecture has only been proved when

$\Delta_1 \leq 2$ [Aigner, Brandt (1993), and Alon, Fischer (1996)],

$\Delta_1 = 3$ and n is huge [Csaba, Shokoufandeh, Szemerédi (2003)].

Near-packing of degree 1 [Eaton (2000)].

G_1 d-degenerate, $\max\{40\Delta_1 \log \Delta_2, 40d\Delta_2\} < n$
[Bollobás, Kostochka, Nakprasit (2008)].

G_1 contains no $K_{2,t}$ and $\Delta_1 > 17t\Delta_2$ [van Batenburg, Kang (2019)].

Bollobás-Eldridge-Catlin Conjecture

Conjecture (Bollobás, Eldridge (1978), & Catlin (1976))
If $(\Delta_1 + 1)(\Delta_2 + 1) \leq n + 1$ *then* G_1 *and* G_2 *pack.*

Theorem (Kaul, Kostochka, Yu (2008))
For $\Delta_1, \Delta_2 \geq 300$,
If $(\Delta_1 + 1)(\Delta_2 + 1) \leq (0.6)n + 1$, *then* G_1 *and* G_2 *pack.*

Theorem (Sauer & Spencer (1978))
If $\Delta_1 \Delta_2 < (0.5)n$, *then* G_1 *and* G_2 *pack.*

Classic Results on Graph Packing

Theorem (Sauer & Spencer (1978))

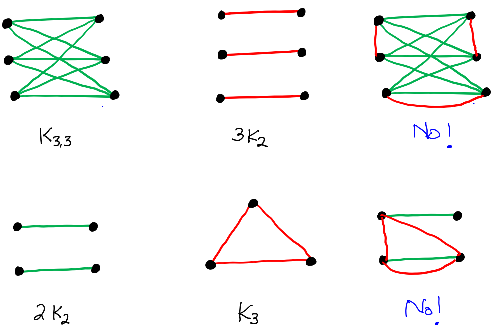
If $2\Delta_1\Delta_2 < n$ *, then* G_1 *and* G_2 *pack.*

Classic Results on Graph Packing

Theorem (Sauer & Spencer (1978))

If $2\Delta_1\Delta_2 < n$, then G_1 and G_2 pack.

- Sharp: $G_1 = \frac{n}{2}K_2$. $G_2 \supseteq K_{\frac{n}{2}+1}$, or $G_2 = K_{\frac{n}{2},\frac{n}{2}}$ with $\frac{n}{2}$ odd.



Classic Results on Graph Packing

Characterization of the extremal graphs for the Sauer-Spencer Theorem.

Theorem (Kaul, Kostochka (2007))

If $2\Delta_1\Delta_2 \leq n$, then
G_1 and G_2 do not pack if and only if
one of G_1 and G_2 is a perfect matching and the other either is $K_{\frac{n}{2},\frac{n}{2}}$ with $\frac{n}{2}$ odd or contains $K_{\frac{n}{2}+1}$.

Classic Results on Graph Packing

Theorem (Brandt (1994))

If G is a graph and T is a tree with $\ell(T)$ leaves, both on n vertices, and $3\Delta(G) + \ell(T) - 2 < n$ then G and T pack.

- A partial step towards the Erdős-Sós conjecture: a graph G contains every tree T with $\ell(T) \leq 3\delta(G) - 2n + 4$.

Classic Results on Graph Packing

Theorem (Brandt (1994))

If G is a graph and T is a tree with $\ell(T)$ leaves, both on n vertices, and $3\Delta(G) + \ell(T) - 2 < n$ then G and T pack.

- Characterization of extremal graphs?

Extremal Graphs for Brandt

Theorem (Brandt (1994))

If G is a graph and T is a tree with $\ell(T)$ leaves, both on n vertices, and $3\Delta(G) + \ell(T) - 2 < n$ then G and T pack.

- Characterization of extremal graphs of Brandt.

Theorem (K., Reiniger (2020+))

If G is a graph and F is a forest, both on n vertices, and $3\Delta(G) + \ell^*(F) \leq n$ then G and F pack unless n is even, $G = \frac{n}{2}K_2$, and $F = K_{1,n-1}$.

- $\ell^*(F) = \ell(F) - 2\,\mathrm{comp}(F)$, where $\mathrm{comp}(F)$ denotes the number of non-trivial components of F.
- $\ell^*(F)$ represents the number of "excess leaves" compared to a linear forest.
- For a tree T, $\ell^*(T) = \ell(T) - 2$.

Extremal Graphs for Brandt

Theorem (Brandt (1994))

If G is a graph and T is a tree with $\ell(T)$ leaves, both on n vertices, and $3\Delta(G) + \ell(T) - 2 < n$ then G and T pack.

- Characterization of extremal graphs of Brandt.

Theorem (K., Reiniger (2020+))

If G is a graph and F is a forest, both on n vertices, and $3\Delta(G) + \ell^*(F) \leq n$ then G and F pack unless n is even, $G = \frac{n}{2}K_2$, and $F = K_{1,n-1}$.

- $\ell^*(F) = \ell(F) - 2\,\mathrm{comp}(F)$, where $\mathrm{comp}(F)$ denotes the number of non-trivial components of F.
- $\ell^*(F)$ represents the number of "excess leaves" compared to a linear forest.
- For a tree T, $\ell^*(T) = \ell(T) - 2$.

A Generalization of Sauer-Spencer & Brandt

- Recall, a graph G is c-degenerate if every subgraph of it has a vertex of degree at most c.
  It is a measure of sparseness of a graph and equivalent to *core number*, or *coloring number*.

A Generalization of Sauer-Spencer & Brandt

Theorem (K., Reiniger (2020+))

Let G be a graph and H a c-degenerate graph, both on n vertices.
Let $d_1^{(G)} \geq d_2^{(G)} \geq \cdots \geq d_n^{(G)}$ *be the degree sequence of G, and similarly for H.*
If $\sum_{i=1}^{\Delta(G)} d_i^{(H)} + \sum_{j=1}^{c} d_j^{(G)} < n$*, then G and H pack.*

A Generalization of Sauer-Spencer & Brandt

Theorem (K., Reiniger (2020+))

Let G be a graph and H a c-degenerate graph, both on n vertices.
Let $d_1^{(G)} \geq d_2^{(G)} \geq \cdots \geq d_n^{(G)}$ *be the degree sequence of G, and similarly for H.*
If $\sum_{i=1}^{\Delta(G)} d_i^{(H)} + \sum_{j=1}^{c} d_j^{(G)} < n$*, then G and H pack.*

- This strengthens Sauer-Spencer, since $c \leq \Delta(H)$.

A Generalization of Sauer-Spencer & Brandt

Theorem (K., Reiniger (2020+))

Let G be a graph and H a c-degenerate graph, both on n vertices.
Let $d_1^{(G)} \geq d_2^{(G)} \geq \cdots \geq d_n^{(G)}$ be the degree sequence of G, and similarly for H.
If $\sum_{i=1}^{\Delta(G)} d_i^{(H)} + \sum_{j=1}^{c} d_j^{(G)} < n$, then G and H pack.

- This also strengthens Brandt's theorem:
  if H is a tree, then $c = 1$, so the second summation is just $\Delta(G)$. For the first summation,

$$\sum_{i=1}^{\Delta(G)} d_i^{(H)} = 2\Delta(G) + \sum_{i=1}^{\Delta(G)} \left(d_i^{(H)} - 2\right) \leq 2\Delta(G) + \ell(H) - 2.$$

A Generalization of Sauer-Spencer & Brandt

Theorem (K., Reiniger (2020+))

Let G be a graph and H a c-degenerate graph, both on n vertices.
Let $d_1^{(G)} \geq d_2^{(G)} \geq \cdots \geq d_n^{(G)}$ be the degree sequence of G, and similarly for H.
If $\sum_{i=1}^{\Delta(G)} d_i^{(H)} + \sum_{j=1}^{c} d_j^{(G)} < n$, then G and H pack.

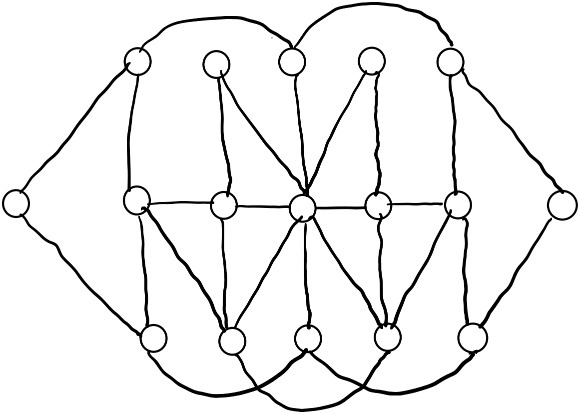
- This Theorem retains all the Sauer-Spencer extremal graphs:
  - $H = \frac{n}{2}K_2$ and $G \supseteq K_{n/2+1}$
  - $H = \frac{n}{2}K_2$ and $G = K_{n/2,n/2}$, with $n/2$ odd
  - $H \supseteq K_{n/2+1}$ and $G = \frac{n}{2}K_2$
  - $H = K_{n/2,n/2}$ and $G = \frac{n}{2}K_2$, with $n/2$ odd

  And it has an additional family of extremal graphs:
  - $H = K_{s,n-s}$ and $G = \frac{n}{2}K_2$, with s odd
    (in particular, $H = K_{1,n-1}$ and $G = \frac{n}{2}K_2$)

  We do not know whether these are all the extremal graphs.

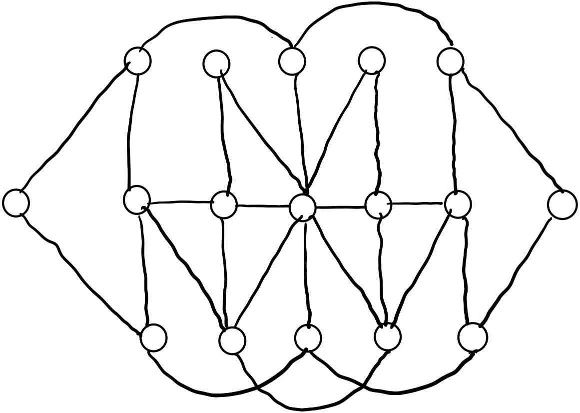
A Packing Puzzle

Can you fill in the numbers $1, 2, \ldots, 17$ in the 17 circles below without repetition so that no two consecutive numbers are placed in circles with a line segment joining them?



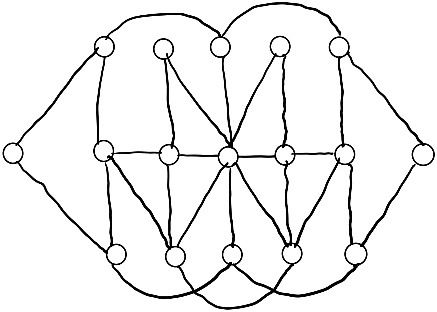
A Packing Puzzle

Can you pack P_{17} with this given graph?



A Packing Puzzle

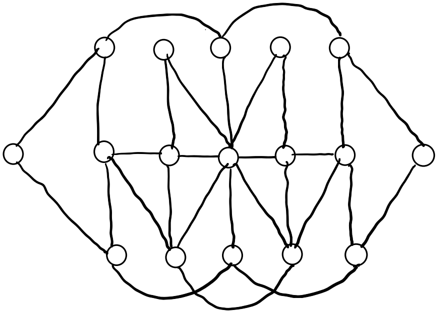
Can you pack P_{17} with this given graph?



- Dirac (If $\Delta(G) \leq n/2 - 1$, then G packs with C_n) fails to apply.

A Packing Puzzle

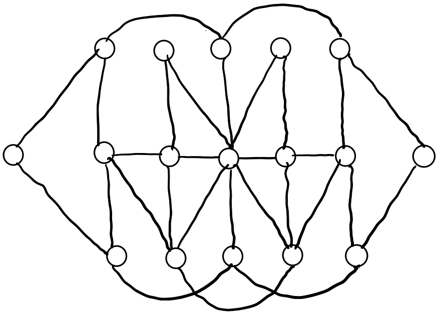
Can you pack P_{17} with this given graph?



- Sauer-Spencer (and its extension) fails to apply.

A Packing Puzzle

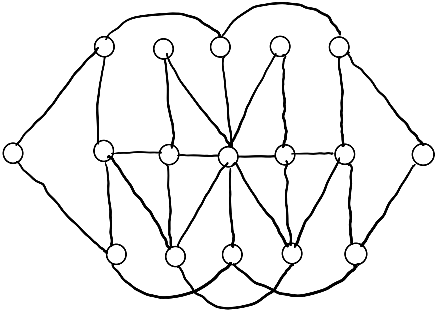
Can you pack P_{17} with this given graph?



- Bollobas-Eldridge-Catlin (if its true) fails to apply.

A Packing Puzzle

Can you pack P_{17} with this given graph?



- Yes! By our result ($G = P_{17}$, H be the given graph which is 2-degenerate, so $c = 2$.)

Some Proof Ideas

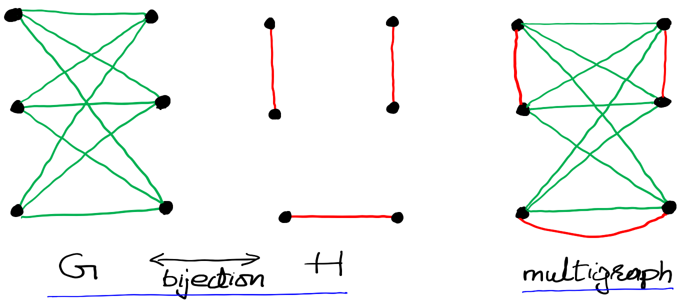
- Structural Analysis of a (hypothetical) minimal counterexample.

Some Proof Ideas

- Think of a bijective mapping $f : V(G) \to V(H)$ as the multigraph with vertices $V(G)$ and edges labelled by G (green) or H (red).

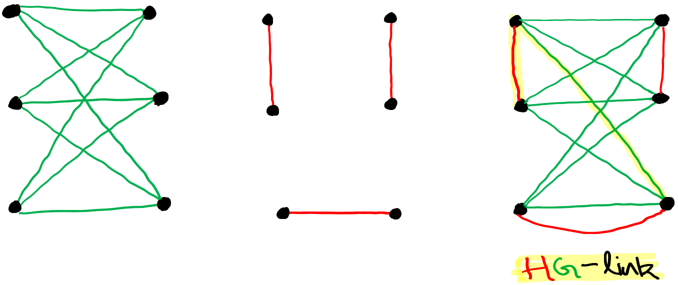
Some Proof Ideas

- Think of a bijective mapping $f : V(G) \to V(H)$ as the multigraph with vertices $V(G)$ and edges labelled by G (green) or H (red).



Some Proof Ideas

- Think of a bijective mapping $f : V(G) \to V(H)$ as the multigraph with vertices $V(G)$ and edges labelled by G (green) or H (red).



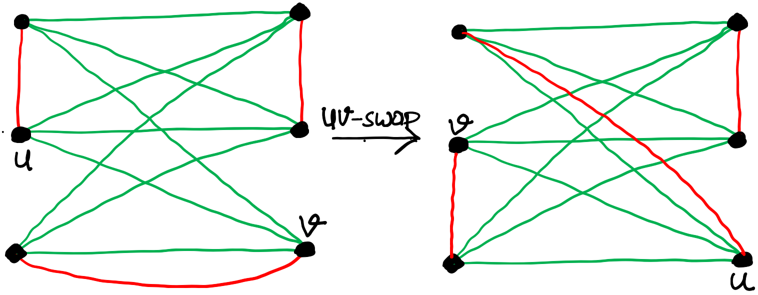
- A link is a copy of P_3 with one G-edge and one H-edge, that is a green-red (or red-green) path.
  We will also say: uv-link, GH-link, etc.

Some Proof Ideas

- From a given mapping f, a uv-swap results in a new mapping f' with $f'(u) = f(v)$, $f'(v) = f(u)$, and $f' = f$ otherwise.
  That is, u and v exchange their green-neighbors.

Some Proof Ideas

- From a given mapping f, a uv-swap results in a new mapping f' with $f'(u) = f(v)$, $f'(v) = f(u)$, and $f' = f$ otherwise.
  That is, u and v exchange their green-neighbors.



Some Proof Ideas

- A quasipacking of G with H is a bijective mapping f whose multigraph is simple except for a single pair of vertices joined by both an G-edge and a H-edge (the conflicting edge).

Outline of the Proof - I

Theorem (K., Reiniger (2020+))

Let G be a graph and H a c-degenerate graph, both on n vertices. Let $d_1^{(G)} \geq d_2^{(G)} \geq \cdots \geq d_n^{(G)}$ be the degree sequence of G, and similarly for H.

If $\sum_{i=1}^{\Delta(G)} d_i^{(H)} + \sum_{j=1}^{c} d_j^{(G)} < n$, then G and H pack.

Outline of the Proof - I

Theorem (K., Reiniger (2020+))

Let G be a graph and H a c-degenerate graph, both on n vertices. Let $d_1^{(G)} \geq d_2^{(G)} \geq \cdots \geq d_n^{(G)}$ be the degree sequence of G, and similarly for H.

If $\sum_{i=1}^{\Delta(G)} d_i^{(H)} + \sum_{j=1}^{c} d_j^{(G)} < n$, then G and H pack.

- Consider a pair of graphs (G, H) satisfying the given condition, with H being c-degenerate, each on n vertices, that do not pack; furthermore assume that H is edge-minimal with this property.
  Thus for any edge e in H, G and $H - e$ pack, and so there is a quasipacking of H and G with conflicting edge e.

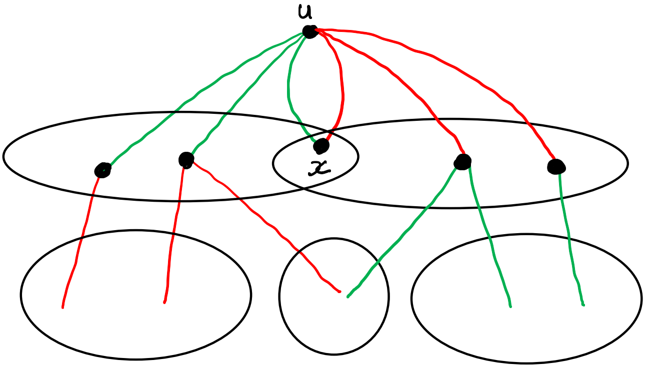
Outline of the Proof - I

- Consider a pair of graphs (G, H) satisfying the given condition, with H being c-degenerate, each on n vertices, that do not pack; furthermore assume that H is edge-minimal with this property.
  Thus for any edge e in H, G and $H - e$ pack, and so there is a quasipacking of H and G with conflicting edge e.

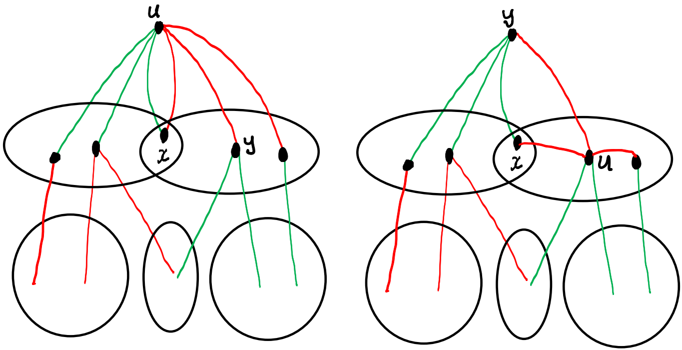
- Let u' be a vertex of minimum positive degree in H, let $x' \in N_H(u')$, and consider a quasipacking f of G with H with conflicting edge $u'x'$. Let $u = f^{-1}(u')$ and $x = f^{-1}(x')$.

Outline of the Proof - I

- Let $\underline{u' \text{ be a vertex of minimum positive degree in } H}$, let $x' \in N_H(u')$, and consider a quasipacking f of G with H with conflicting edge $u'x'$. Let $u = f^{-1}(u')$ and $x = f^{-1}(x')$.

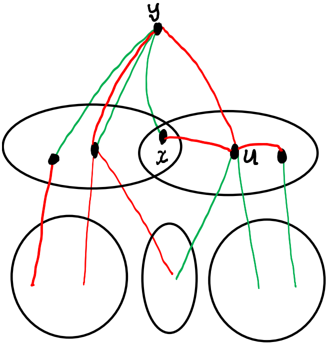


Outline of the Proof - I



- There is a uy-link for every $y \in V(G) \setminus \{u, x\}$.
  Perform a uy-swap: since G and H do not pack, there must be some conflicting edge, and such a conflict must involve an H-edge incident to either u or y. In either case, this along with the conflicting G-edge gives a uy-link in the original multigraph.

Outline of the Proof - I



- There is a uy-link for every $y \in V(G) \setminus \{u, x\}$.
  Perform a uy-swap: since G and H do not pack, there must be some conflicting edge, and such a conflict must involve an H-edge incident to either u or y. In either case, this along with the conflicting G-edge gives a uy-link in the original multigraph.

Outline of the Proof - I

- There is a uy-link for every $y \in V(G) \setminus \{u, x\}$.
- There are two links from u to itself, using the parallel edges ux in each order. Thus, there are at least n links from u.

Outline of the Proof - I

- There are at least n links from u.
- The number of GH-links from u is at most $\sum_{y \in N_G(u)} \deg_H(f(y))$.
  (sum of red-degrees of green neighbors of u)
- The number of HG-links from u is at most
  $$\sum_{z' \in N_H(u')} \deg_G(f^{-1}(z')).$$
  (sum of green-degrees of red neighbors of u)

Outline of the Proof - I

n

$\leq$ # links from u

$$\leq \sum_{y \in N_G(u)} \deg_H(f(y)) + \sum_{z' \in N_H(u')} \deg_G(f^{-1}(z'))$$

$$\leq \sum_{i=1}^{\Delta(G)} d_i^{(H)} + \sum_{j=1}^{c} d_j^{(G)}$$, by the choice of u'

Contradiction!

Outline of the Proof - II

Theorem (K., Reiniger (2020+))

If G is a graph and F is a forest, both on n vertices, and $3\Delta(G) + \ell^*(F) \leq n$ then G and F pack unless n is even, $G = \frac{n}{2}K_2$, and $F = K_{1,n-1}$.

- Now, we suppose that H is a forest, henceforth called F, and that $3\Delta(G) + \ell^*(F) = n$.
  We still assume that G and F do not pack, and that F is edge-minimal with this property.
- If $\Delta(G) = 1$, then it is easy to show that n is even, $G = \frac{n}{2}K_2$, and $F = K_{1,n-1}$.
  So we can assume that $\Delta(G) > 1$, and seek a contradiction.

Outline of the Proof - II

- In the current setup, u' is a leaf of F and x' its neighbor.

$$
\begin{aligned}
n &\le \#\text{ links from } u \\
&\le \sum_{y \in N_G(u)} \deg_F(f(y)) + \deg_G(x) && (1)\\
&\le \sum_{y \in N_G(u)} \big(\deg_F(f(y)) - 2\big) + 2\Delta(G) + \Delta(G) && (2)\\
&\le \sum_{y \in N_G(u)} \max\{\deg_F(f(y)) - 2, 0\} + 3\Delta(G) && (3)\\
&\le \sum_{i=1}^{n} \max\{d_i^{(F)} - 2, 0\} + 3\Delta(G) = 3\Delta(G) + \ell^*(F) = n, && (4)
\end{aligned}
$$

so we have equality throughout.

Outline of the Proof - II

- Analyzing each of the four equations above, gives us:

Lemma

For any leaf u' of F and x' its neighbor, and a quasipacking f of G with F with $f(u) = u'$ and $f(x) = x'$ and conflicting edge ux, we have the following.

1. *For every $y \in V(G) \setminus \{u, x\}$, there is a unique link from u to y; there is no link from u to x; and there are two links from u to itself.*
2. $\deg_G(x) = \deg_G(u) = \Delta(G)$.
3. *For every $w \in N_G(u)$, $\deg_F(f(w)) \geq 2$.*
4. *For every $w \notin N_G(u)$, $\deg_F(f(w)) \leq 2$.*

Outline of the Proof - II

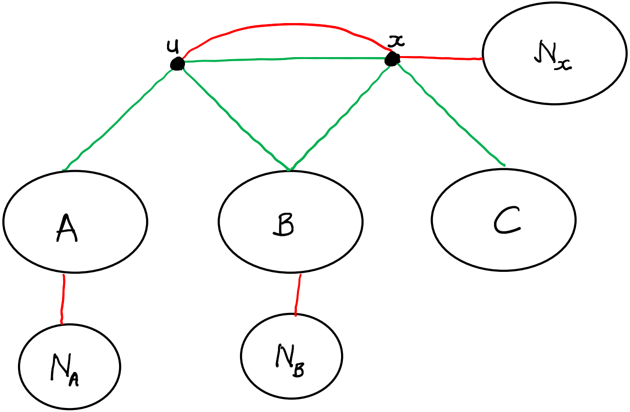
Lemma

For any leaf u' of F and x' its neighbor, and a quasipacking f of G with F with $f(u) = u'$ and $f(x) = x'$ and conflicting edge ux, we have the following.

1. $N_G[u] = N_G[x]$.
2. $G[N_G[u]]$ *is a clique component.*

Use appropriately chosen swap operations and the previous lemma to show that the structure of quasipacking looks like:

Outline of the Proof - II



- We can show that
  - $\{u, x\}, A, B, C, N_A, N_B, N_x$ is a partition of $V(G)$
  - A, N_A, N_C, C are all empty
  - $N_G[u] = N_G[x] = \{u, x\} \cup B$ forms a clique in G.

Outline of the Proof - II

- Let $G[Q]$ be the clique component of G given by the Lemma 2.
  Let z be a vertex of Q with smallest F-degree larger than 1 (such a choice is possible by Lemma 1).
  Let $z_1, z_2 \in V(G)$ be two F-neighbors of z.

Outline of the Proof - II

- Let $G[Q]$ be the clique component of G given by the Lemma 2. Let z be a vertex of Q with smallest F-degree larger than 1 (such a choice is possible by Lemma 1).
  Let $z_1, z_2 \in V(G)$ be two F-neighbors of z.

- We can show that $Q \cup \{z_1, z_2\} \setminus \{u, z\}$ is F-independent.
- Let $X = f(Q \cup \{z_1, z_2\} \setminus \{u, z\})$.
  Let $g : V(G) \to V(F)$ be a bijection such that $g(Q) = X$.
  Since $G[Q]$ is a clique component and X is independent, g is a packing if and only if $g|_{G-Q}$ is a packing of $G - Q$ with $F - X$.
- Since G and F do not pack, we must have that $G - Q$ and $F - X$ do not pack.
  We get a contradiction by showing that $G - Q$ and $F - X$ pack.

Thank You!
Questions?

Conjecture (Erdős-Sós (1962))

Let G be a graph of order n and T be a tree of size k. If $e(G) < \frac{1}{2}n(n-k)$ then T and G pack.

Conjecture (Bollobás-Eldridge (1978), & Catlin (1976))

If $(\Delta_1 + 1)(\Delta_2 + 1) \leq n + 1$ then G_1 and G_2 pack.

- Characterize all extremal graphs of:

Theorem (K., Reiniger (2020+))

Let G be a graph and H a c-degenerate graph, both on n vertices. Let $d_1^{(G)} \geq d_2^{(G)} \geq \cdots \geq d_n^{(G)}$ be the degree sequence of G, and similarly for H.

If $\sum_{i=1}^{\Delta(G)} d_i^{(H)} + \sum_{j=1}^{c} d_j^{(G)} < n$, then G and H pack.

Thank You!
Questions?

Conjecture (Erdős-Sós (1962))

Let G be a graph of order n and T be a tree of size k.
If $e(G) < \frac{1}{2}n(n-k)$ then T and G pack.

Conjecture (Bollobás-Eldridge (1978), & Catlin (1976))

If $(\Delta_1 + 1)(\Delta_2 + 1) \leq n + 1$ then G_1 and G_2 pack.

- Characterize all extremal graphs of:

Theorem (K., Reiniger (2020+))

Let G be a graph and H a c-degenerate graph, both on n vertices. Let $d_1^{(G)} \geq d_2^{(G)} \geq \cdots \geq d_n^{(G)}$ be the degree sequence of G, and similarly for H.
If $\sum_{i=1}^{\Delta(G)} d_i^{(H)} + \sum_{j=1}^{c} d_j^{(G)} < n$, then G and H pack.