Graph Packing – Conjectures and Results

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A graph $G$ is a tuple $(V(G), E(G))$, where $V(G)$ is a set of elements called vertices, and $E(G)$ is a collection of sets, each consisting of two elements of $V(G)$, called edges.

Graphs represent (symmetric) binary relations on an underlying set.

**Notation:** Some basic parameters

- Order of $G$, $n(G) = |V(G)|$, the number of vertices in $G$.
- Size of $G$, $e(G) = |E(G)|$, the number of edges in $G$. 
Graph Theoretic Notation

Notation: Elementary families of graphs

- \( K_n \), complete graph or clique. 
  - \( n \) vertices, and all \( \binom{n}{2} \) edges present.

- \( K_{m,n} \), complete bipartite graph or biclique. 
  - \( V(G) = V_1 \sqcup V_2 \), with \( |V_1| = m \), \( |V_2| = n \). No edges within \( V_1 \) or \( V_2 \), and all \( mn \) edges between \( V_1 \) and \( V_2 \).

- \( mK_2 \), perfect matching. \( m \) disjoint edges.
Notation: Another basic parameter

- Maximum degree of $G$, $\Delta(G)$, the maximum number of edges incident to any vertex of $G$.
  - $\Delta(K_n) = n - 1$
  - $\Delta(K_{m,n}) = \max\{m, n\}$
  - $\Delta(mK_2) = 1$

- Minimum degree of $G$, $\delta(G)$, the minimum number of edges incident to any vertex of $G$. 
Let $G_i = (V_i, E_i)$ be graphs of order at most $n$, with maximum degree $\Delta_i$, $i = 1, \ldots, k$. 
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$G_1, \ldots, G_k$ are said to pack if there exist injective mappings of the vertex sets into $[n]$, $V_i \rightarrow [n] = \{1, 2, \ldots, n\}$, $i = 1, \ldots, k$ such that the images of the edge sets do not intersect.
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We may assume $|V_i| = n$ by adding isolated vertices.
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Understanding how to pack two graphs ($k = 2$) is typically the most crucial case.
Introduction

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$G_1, G_2$ are said to pack if

- there exists a bijection $V_1 \leftrightarrow V_2$ such that $e \in E_1 \Rightarrow e \not\in E_2$.
- $G_1$ is a subgraph of $\overline{G_2}$.
Examples and Non-Examples

C₅

C₅
Examples and Non-Examples

$C_5$

$C_5$

Packing
Examples and Non-Examples

C₅

C₅

Packing

K₄,₄

4 K₂
Examples and Non-Examples

C₅

C₅

Packing

K₄,₄

4 K₂

Packing
Examples and Non-Examples

K₃,₃

3 K₂
Examples and Non-Examples

$K_{3,3}$

$3 \cdot K_2$

No Packing
Examples and Non-Examples

\[ K_{3,3} \]

\[ 2 \text{ } K_2 \]

\[ 3 \text{ } K_2 \]

\[ \text{No Packing} \]

\[ K_3 \]
Examples and Non-Examples

$K_{3,3}$

$3K_2$

$2K_2$

$K_3$

No Packing

No Packing

Graph Packing – p.5/26
Hamiltonian Cycle in graph $G$ : A cycle through all the vertices in $G$.
Whether the $n$-cycle $C_n$ packs with $\overline{G}$.
A Common Generalization

- Existence of a subgraph $H$ in $G$: $H$ is a subgraph of $G$. Whether $H$ packs with $\bar{G}$. 
A Common Generalization

- **Existence of a subgraph \( H \) in \( G \):** \( H \) is a subgraph of \( G \). Whether \( H \) packs with \( \overline{G} \).

- **Equitable \( k \)-coloring of graph \( G \):** A proper \( k \)-coloring of \( G \) such that sizes of all color classes differ by at most 1. A partition of \( V(G) \) into \( k \) classes of cardinality \( \lfloor \frac{n}{k} \rfloor \) or \( \lceil \frac{n}{k} \rceil \), s.t. there is no edge between two vertices in the same class. Whether \( G \) packs with \( k \) cliques of order \( n/k \).
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- **Existence of a subgraph** $H$ in $G$: $H$ is a subgraph of $G$. Whether $H$ packs with $\overline{G}$.

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- **Turán-type problems**: $ex(n, H) =$ Maximum size of any $n$-vertex graph not containing $H$ as a subgraph. Every graph with at least $ex(n, H)$ edges packs with $\overline{H}$.
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- **Existence of a subgraph** $H$ in $G$: $H$ is a subgraph of $G$. Whether $H$ packs with $\overline{G}$.

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- **Ramsey-type problems**.
A Common Generalization

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- **Equitable \( k \)-coloring of graph** \( G \) : A proper \( k \)-coloring of \( G \) such that sizes of all color classes differ by at most 1. A partition of \( V(G) \) into \( k \) classes of cardinality \( \lfloor \frac{n}{k} \rfloor \) or \( \lceil \frac{n}{k} \rceil \), s.t. there is no edge between two vertices in the same class. Whether \( G \) packs with \( k \) cliques of order \( n/k \).

- **Turán-type problems** : \( ex(n, H) = \) Maximum size of any \( n \)-vertex graph not containing \( H \) as a subgraph. Every graph with at least \( ex(n, H) \) edges packs with \( \overline{H} \).

- **Ramsey-type problems**.

- **“most” problems in Extremal Graph Theory**.
In packing problems, each member of a ‘large’ family of graphs contains each member of another ‘large’ family of graphs.
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In subgraph problems, (usually) at least one of the two graphs is fixed.
In **packing problems**, each member of a ‘large’ family of graphs contains each member of another ‘large’ family of graphs.
Some Examples

In **packing problems**, each member of a ‘large’ family of graphs contains each member of another ‘large’ family of graphs.

**Erdős-Sos Conjecture (1963)**:
Let $G$ be a graph of order $n$ and $T$ be a tree of size $k$. If $e(G) < \frac{1}{2}n(n - k)$ then $T$ and $G$ pack.

Each graph with at least $\frac{1}{2}n(k - 1)$ edges contains every tree of size $k$.

Sharp, if true. Take disjoint copies of $k$-cliques.

Known only for special classes of trees, graphs of large girth, etc.
In packing problems, each member of a ‘large’ family of graphs contains each member of another ‘large’ family of graphs.

**Theorem:** If $e(G_1)e(G_2) < \binom{n}{2}$, then $G_1$ and $G_2$ pack.

**Proof.** Pick a random bijection between $V(G_1)$ and $V(G_2)$, uniformly among the set of all $n!$ such bijections. Now, show the probability that this (random) bijection does not give a packing of $G_1$ and $G_2$ is $< 1$. 
Some Examples

In packing problems, each member of a ‘large’ family of graphs contains each member of another ‘large’ family of graphs.

**Theorem**: If \( e(G_1) < n - 1 \) and \( e(G_2) < n - 1 \), then \( G_1 \) and \( G_2 \) pack.

Note that a tree on \( n \) vertices contains exactly \( n - 1 \) edges.
In **packing problems**, each member of a ‘large’ family of graphs contains each member of another ‘large’ family of graphs.

**Theorem**: If $e(G_1) < n - 1$ and $e(G_2) < n - 1$, then $G_1$ and $G_2$ pack.

**Theorem** [Bollobas + Eldridge, 1978, & Teo + Yap, 1990]: If $\Delta_1, \Delta_2 < n - 1$, and $e(G_1) + e(G_2) \leq 2n - 2$, then $G_1$ and $G_2$ do not pack if and only if they are one of the finitely many specified pairs of graphs.
Gyárfás Tree Packing Conjecture (1976):
Let $T_i$ denote a tree of order $i$.
Then, any trees $T_2, \ldots, T_n$ can be packed.

In fact, this gives a decomposition of $K_n$ into any $T_2, \ldots, T_n$.

Known only for special classes of trees, shorter sequences of trees, etc.
Sauer and Spencer’s Packing Theorem

Theorem [Sauer + Spencer, 1978]:
If $2\Delta_1 \Delta_2 < n$, then $G_1$ and $G_2$ pack.
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If $2\Delta_1 \Delta_2 < n$, then $G_1$ and $G_2$ pack.

If $\delta(G) > \frac{(2k-1)(n-1)+1}{2k}$, then $G$ contains all graphs with maximum degree at most $k$. 
Theorem [Sauer + Spencer, 1978]:
If \(2\Delta_1\Delta_2 < n\), then \(G_1\) and \(G_2\) pack.

This is sharp.

For \(n\) even.

\(G_1 = \frac{n}{2}K_2\), a perfect matching on \(n\) vertices.

\(G_2 \supseteq K_{\frac{n}{2}+1}\), or

\(G_2 = K_{\frac{n}{2}, \frac{n}{2}}\) with \(\frac{n}{2}\) odd.

Then, \(2\Delta_1\Delta_2 = n\), and \(G_1\) and \(G_2\) do not pack.
Sauer and Spencer’s Packing Theorem

\[ G_1 = K_{\frac{n}{2}, \frac{n}{2}} \text{ with } \frac{n}{2} \text{ odd} \]

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\[ K_{3,3} \]

\[ 3 K_2 \]

No Packing

\[ 2 K_2 \]

\[ K_3 \]

No Packing
Theorem 1 [Kaul + Kostochka, 2005]:
If $2\Delta_1 \Delta_2 \leq n$, then $G_1$ and $G_2$ do not pack if and only if one of $G_1$ and $G_2$ is a perfect matching and the other either is $K_{\frac{n}{2}, \frac{n}{2}}$ with $\frac{n}{2}$ odd or contains $K_{\frac{n}{2}+1}$.

This result characterizes the extremal graphs for the Sauer-Špencer Theorem.

To appear in *Combinatorics, Probability and Computing*. 
Theorem 1 [Kaul + Kostochka, 2005]:
If $\Delta_1 \Delta_2 \leq \frac{1}{2}n$, then
$G_1$ and $G_2$ do not pack if and only if
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either is $K_{\frac{n}{2}, \frac{n}{2}}$ with $\frac{n}{2}$ odd or contains $K_{\frac{n}{2}+1}$.

$\Delta_1 \Delta_2 \leq \frac{1}{2} n$ is sharp exactly when one of $\Delta_1$, $\Delta_2$ is small.

Can we improve the bound on $\Delta_1 \Delta_2$, if both $\Delta_1$ and $\Delta_2$ are large?
Theorem 1 [Kaul + Kostochka, 2005]:
If $\Delta_1 \Delta_2 \leq \frac{1}{2} n$, then
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either is $K_{\frac{n}{2}, \frac{n}{2}}$ with $\frac{n}{2}$ odd or contains $K_{\frac{n}{2}+1}$.

Bollobás-Eldridge Graph Packing Conjecture:
If $(\Delta_1 + 1)(\Delta_2 + 1) \leq n + 1$ then $G_1$ and $G_2$ pack.

Theorem 1 can be thought of as a small step towards
this longstanding conjecture.
Bollobás-Eldridge Graph Packing Conjecture [1978]:
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If \((\Delta_1 + 1)(\Delta_2 + 1) \leq n + 1\) then \(G_1\) and \(G_2\) pack.

If \(\delta(G) > \frac{kn-1}{k+1}\), then 
\(G\) contains all graphs with maximum degree at most \(k\).
Bollobás-Eldridge Graph Packing Conjecture [1978]:
If \((\Delta_1 + 1)(\Delta_2 + 1) \leq n + 1\) then \(G_1\) and \(G_2\) pack.

If true, this conjecture would be sharp.

\[ n = (d_1 + 1)(d_2 + 1) - 2, \quad \Delta_1 = d_1, \quad \Delta_2 = d_2. \]
Bollobás-Eldridge Graph Packing Conjecture [1978]:
If \((\Delta_1 + 1)(\Delta_2 + 1) \leq n + 1\) then \(G_1\) and \(G_2\) pack.

If true, this conjecture would be a considerable extension of the Hajnal-Szemerédi theorem on equitable colorings:

Every graph \(G\) has an equitable \(k\)-coloring for \(k \geq \Delta(G) + 1\).

Equitable colorings of graphs have been used to

- extend Chernoff-Hoeffding concentration bounds to dependent random variables (Pemmaraju, 2003)
- extend Arnold-Groeneveld order statistics bounds to dependent random variables (Kaul + Jacobson, 2005)
Bollobás-Eldridge Graph Packing Conjecture [1978]:
If \((\Delta_1 + 1)(\Delta_2 + 1) \leq n + 1\) then \(G_1\) and \(G_2\) pack.

The conjecture has only been proved when
\(\Delta_1 \leq 2\) [Aigner + Brandt (1993), and Alon + Fischer (1996)],
\(\Delta_1 = 3\) and \(n\) is huge [Csaba + Shokoufandeh + Szemerédi (2003)].

One of the graphs is sparse (\(d\)-degenerate) [Bollobás + Kostochka + Nakprasit (2004)].

Near-packing of degree 1 [Eaton (2000)].
Let us consider a refinement of the Bollobás-Eldridge Conjecture.

**Conjecture**: For a fixed $0 \leq \epsilon \leq 1$. If $(\Delta_1 + 1)(\Delta_2 + 1) \leq \frac{n}{2} (1 + \epsilon) + 1$, then $G_1$ and $G_2$ pack.

For $\epsilon = 0$, this is essentially the Sauer-Spencer Theorem, while $\epsilon = 1$ gives the Bollobás-Eldridge conjecture.

For any $\epsilon > 0$ this would improve the Sauer-Spencer result (in a different way than Theorem 1).
Theorem 2 [Kaul + Kostochka + Yu, 2005+]:
For $\epsilon = 0.2$, and $\Delta_1, \Delta_2 \geq 300$,
If $(\Delta_1 + 1)(\Delta_2 + 1) \leq \frac{n}{2}(1 + \epsilon) + 1$, then $G_1$ and $G_2$ pack.
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In other words,

Theorem 2 [Kaul + Kostochka + Yu, 2005+]:
For $\Delta_1, \Delta_2 \geq 300$,
If $(\Delta_1 + 1)(\Delta_2 + 1) \leq (0.6)n + 1$, then $G_1$ and $G_2$ pack.
Some Ideas for the Proofs

We have to analyze the ‘minimal’ graphs that do not pack (under the given condition on $\Delta_1$ and $\Delta_2$).

$(G_1, G_2)$ is a **critical pair** if $G_1$ and $G_2$ do not pack, but for each $e_1 \in E(G_1)$, $G_1 - e_1$ and $G_2$ pack, and for each $e_2 \in E(G_2)$, $G_1$ and $G_2 - e_2$ pack.

$G_1$ and $G_2$ do not pack, but removing one edge from either $G_1$ or $G_2$ allows them to pack.
Some Ideas for the Proofs

Each bijection \( f : V_1 \to V_2 \) generates a (multi)graph \( G_f \), with

\[
V(G_f) = \{(u, f(u)) : u \in V_1\}
\]

\((u, f(u)) \leftrightarrow (u', f(u')) \Leftrightarrow uu' \in E_1 \text{ or } f(u)f(u') \in E_2\)

Every vertex has two kinds of neighbors:
- green from \( G_1 \)
- red from \( G_2 \).
Some Ideas for the Proofs

Each bijection $f : V_1 \rightarrow V_2$ generates a (multi)graph $G_f$, with

$$V(G_f) = \{(u, f(u)) : u \in V_1\}$$

$$(u, f(u)) \leftrightarrow (u', f(u')) \iff uu' \in E_1 \text{ or } f(u)f(u') \in E_2$$

Every vertex has two kinds of neighbors: green from $G_1$ and red from $G_2$. 

\[ \begin{array}{c}
\text{G}_1 \\
\text{G}_2 \\
\text{G}_f \\
gives
\end{array} \]
\( (u_1, \ldots, u_k)\)-switch means replace \( f \) by \( f' \), with

\[
f'(u) = \begin{cases} 
  f(u) , & u \neq u_1, u_2, \ldots, u_k \\
  f(u_{i+1}) , & u = u_i , 1 \leq i \leq k - 1 \\
  f(u_1) , & u = u_k
\end{cases}
\]
Some Ideas for the Proofs

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    f(u_1) & u = u_k
  \end{cases}
\end{align*}
\]

green-neighbors of \(u_i \rightarrow\) green-neighbors of \(u_{i-1}\)
$(u_1, \ldots, u_k)$-switch means replace $f$ by $f'$, with

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  f(u_1) & , \quad u = u_k
\end{cases}
\]

\[
G_f \quad \text{(u_1, u_2)-switch} \quad G_{f'}
\]
An important structure that we utilize in our proof is -

\[(u_1, u_2; 1, 2)-\text{link}\] is a path of length two (in \(G_f\)) from \(u_1\) to \(u_2\) whose first edge is in \(E_1\) and the second edge is in \(E_2\).

A green-red path of length two from \(u_1\) to \(u_2\).
Some Ideas for the Proofs

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A green-red path of length two from \(u_1\) to \(u_2\).

For \(e \in E_1\), an \(e\)-packing (quasi-packing) of \((G_1, G_2)\) is a bijection \(f\) between \(V_1\) and \(V_2\) such that \(e\) is the only edge in \(E_1\) that shares its incident vertices with an edge from \(E_2\).

Such a packing exists for every edge \(e\) in a critical pair.
Outline of the Proof of Theorem 2

Key Lemma: Let $u_1, \ldots, u_k$ be vertices of $G$. If

- for any $i$, there is no red-green path from $u_i$ to $u_{i+1}$, and
- for $1 \leq i < j \leq k$, if $u_i u_j$ is a red edge, then $u_{i+1} u_{j+1}$ is either a red edge or is not an edge.

then a $(u_1, \ldots, u_k)$-switch does not create new conflicting edges.
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then a \((u_1, \ldots, u_k)\)-switch does not create new conflicting edges.
Using the Key Lemma

Consider a critical pair \((G_1, G_2)\).

There is a bijection between \(V(G_1)\) and \(V(G_2)\) with exactly one conflicting edge.

Why is the Key Lemma useful?
Using the Key Lemma

Why is the Key Lemma useful?
Structure of Counterexamples I

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Structure of Counterexamples I

A = vertices with only green-red paths from \( u^* \)

\( A = \frac{n}{2} (1 - \epsilon) \)

B = vertices with only red-green paths from \( u^* \)

\( B = \frac{n}{2} (1 - \epsilon) \)

C = vertices with both types of paths from \( u^* \)

\( C \leq n\epsilon \)
No red-green paths from $u^*$ to $A$.  
No green-red paths from $u^*$ to $B$.  

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Unique red-green paths from $A$ to $B$.

$$(u^*, a, c, b) - \text{switch}$$
Let $N$ be the number of pairs of vertices in $A \times B$ with exactly one red-green path between them.
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**Lower Bound on $N$**: $|A||B| - |A|\left(\Delta_1\Delta_2 - |B|\right)$, a counting argument.

**Upper Bound on $N$**: $M\Delta_1\Delta_2$, where $M$ is the number of central vertices on the unique red-green paths.
The Primary Inequality

Let \( N \) be the number of pairs of vertices in \( A \times B \) with exactly one red-green path between them.

**Lower Bound on** \( N \) \( : |A| |B| - |A|\( (\Delta_1 \Delta_2 - |B|) \)\), a counting argument.

**Upper Bound on** \( N \) \( : M \Delta_1 \Delta_2 \), where \( M \) is the number of central vertices on the unique red-green paths.

Compare the lower bound and the upper bound of \( N \).

\[
|A| |B| - |A|(\Delta_1 \Delta_2 - |B|) \leq M \Delta_1 \Delta_2
\]

Get an inequality for \( \epsilon \), leading to a contradiction.

**Need** an upper bound on \( M \) !
The Key Lemma –

**Lemma 1**: Let \((G_1, G_2)\) be a critical pair and \(2\Delta_1\Delta_2 \leq n\). Given any \(e \in E_1\), in a \(e\)-packing of \((G_1, G_2)\) with \(e = u_1u'_1\), the following statements are true.

(i) For every \(u \neq u'_1\), there exists either a unique \((u_1, u; 1, 2)\)-link or a unique \((u_1, u; 2, 1)\)-link,

(ii) there is no \((u_1, u'_1; 1, 2)\)-link or \((u_1, u'_1; 2, 1)\)-link,

(iii) \(2\deg_{G_1}(u_1)\deg_{G_2}(u_1) = n\).
The Key Lemma –

**Lemma 1**: Let \((G_1, G_2)\) be a critical pair and \(2\Delta_1\Delta_2 \leq n\). Given any \(e \in E_1\), in an \(e\)-packing of \((G_1, G_2)\) with \(e = u_1u_1\), the following statements are true.

(i) For every \(u \neq u'_1\), there exists either a unique \((u_1, u; 1, 2)\)-link or a unique \((u_1, u; 2, 1)\)-link,

(ii) there is no \((u_1, u'_1; 1, 2)\)-link or \((u_1, u'_1; 2, 1)\)-link,

(iii) \(2\deg_{G_1}(u_1)\deg_{G_2}(u_1) = n\).
The Key Lemma –

**Lemma 1**: Let \((G_1, G_2)\) be a critical pair and \(2\Delta_1 \Delta_2 \leq n\). Given any \(e \in E_1\), in a \(e\)-packing of \((G_1, G_2)\) with \(e = u_1 u'_1\), the following statements are true.

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(iii) \(2\deg_{G_1}(u_1)\deg_{G_2}(u_1) = n\).
Lemma 2: If $2\Delta_1\Delta_2 = n$ and $(G_1, G_2)$ is a critical pair, then every component of $G_i$ is either $K_{\Delta_i,\Delta_i}$ with $\Delta_i$ odd, or an isolated vertex, or $K_{\Delta_i+1}$, $i = 1, 2$.

Lemma 2 allows us to settle the case of: $\Delta_1$ or $\Delta_2 = 1$.

If $\Delta_2 = 1$, i.e., $G_2$ is a matching. Then $\Delta_1 = \frac{n}{2}$.

If $G_1$ contains $K_{\Delta_1,\Delta_1}$, then simply $G_1 = K_{\frac{n}{2}, \frac{n}{2}}$. $K_{\frac{n}{2}, \frac{n}{2}}$ cannot pack with a matching iff the matching is perfect and $\frac{n}{2}$ is odd.

If $G_1$ consists of $K_{\frac{n}{2}+1}$ and $\frac{n}{2} - 1$ isolated vertices, then it does not pack with a matching iff the matching is perfect.
Outline of the Proof of Theorem 1

Now, we have to give a packing for all remaining pairs of graphs, to eliminate their possibility.

The following Lemma says $K_{\Delta_1, \Delta_1}$ exists only when $K_{\Delta_2, \Delta_2}$ does, and vice-versa.

**Lemma 3**: Let $\Delta_1, \Delta_2 > 1$ and $2\Delta_1\Delta_2 = n$. If $(G_1, G_2)$ is a critical pair and the conflicted edge in a quasi-packing belongs to a component $H$ of $G_2$ isomorphic to $K_{\Delta_2, \Delta_2}$, then every component of $G_1$ sharing vertices with $H$ is $K_{\Delta_1, \Delta_1}$.

Now, we pack such graphs.
Lemma 4: Suppose that $\Delta_1, \Delta_2 > 1$ and odd, and $2\Delta_1\Delta_2 = n$. If $G_1$ consists of $\Delta_2$ copies of $K_{\Delta_1, \Delta_1}$ and $G_2$ consists of $\Delta_1$ copies of $K_{\Delta_2, \Delta_2}$, then $G_1$ and $G_2$ pack.
Outline of the Proof of Theorem 1

Lets eliminate the only remaining possibility.

**Lemma 5**: Let $\Delta_1, \Delta_2 > 1$ and $2\Delta_1 \Delta_2 = n$. If every non-trivial component of $G_i$ is $K_{\Delta_i+1}$, $i = 1, 2$, then $G_1$ and $G_2$ pack.

This would complete the proof of Theorem 1.