New Results on Graph Packing

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$G_1$ and $G_2$ are said to pack if there exist injective mappings of the vertex sets into $[n]$, $V_i \to [n] = \{1, 2, \ldots, n\}$, $i = 1, 2$, such that the images of the edge sets do not intersect.
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We may assume $|V_1| = |V_2| = n$ by adding isolated vertices.
Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be graphs of order at most $n$, with maximum degree $\Delta_1$ and $\Delta_2$, respectively.

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- there exists a bijection $V_1 \leftrightarrow V_2$ such that $e \in E_1 \Rightarrow e \notin E_2$.
- $G_1$ is a subgraph of $\overline{G_2}$. 
Examples and Non-Examples

C_5

C_5
Examples and Non-Examples

$C_5$

$C_5$

Packing
Examples and Non-Examples

$C_5$

$C_5$

$K_{4,4}$

$4K_2$

Packing
Examples and Non-Examples

C₅

C₅

Packing

K₄,₄

4 K₂

Packing
Examples and Non-Examples

\[ K_{3,3} \quad \text{3} \, K_2 \]
Examples and Non-Examples

\[ K_{3,3} \]

\[ 3 \, K_2 \]

No Packing
Examples and Non-Examples

- $K_{3,3}$
- 3 $K_2$
- No Packing
- 2 $K_2$
- $K_3$
Examples and Non-Examples

$K_{3,3}$

3 $K_2$

No Packing

2 $K_2$

$K_3$

No Packing
Hamiltonian Cycle in graph $G$: Whether the $n$-cycle $C_n$ packs with $\overline{G}$. 

A Common Generalization
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- Turan-type problems (forbidden subgraphs).
- Ramsey-type problems.
- “most” problems in Extremal Graph Theory.
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A Distinction

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In subgraph problems, (usually) at least one of the two graphs is fixed.

Erdős-Sos Conjecture: Let $G$ be a graph of order $n$ and $T$ be a tree of size $k$. If $e(G) < \frac{1}{2}n(n - k)$ then $T$ and $G$ pack.
Theorem [Sauer + Spencer, 1978]:
If $2\Delta_1\Delta_2 < n$, then $G_1$ and $G_2$ pack.
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If $2\Delta_1\Delta_2 < n$, then $G_1$ and $G_2$ pack.

This is sharp.

For $n$ even.

$G_1 = \frac{n}{2}K_2$, a perfect matching on $n$ vertices.

$G_2 \supseteq K_{\frac{n}{2}+1}$, or

$G_2 = K_{\frac{n}{2}, \frac{n}{2}}$ with $\frac{n}{2}$ odd.

Then, $2\Delta_1\Delta_2 = n$, and $G_1$ and $G_2$ do not pack.
Sauer and Spencer’s Packing Theorem

\[ G_1 = K_{\frac{n}{2}, \frac{n}{2}} \text{ with } \frac{n}{2} \text{ odd} \]

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Theorem 1 [Kaul + Kostochka, 2005]:
If $2\Delta_1 \Delta_2 \leq n$, then
$G_1$ and $G_2$ do not pack if and only if
one of $G_1$ and $G_2$ is a perfect matching and the other
either is $K_{\frac{n}{2}, \frac{n}{2}}$ with $\frac{n}{2}$ odd or contains $K_{\frac{n}{2}+1}$.

This result characterizes the extremal graphs for the
Sauer-Spencer Theorem.

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This result can also be thought of as a small step
towards the well-known Bollobás-Eldridge conjecture.

Bollobás-Eldridge Graph Packing Conjecture:
If $(\Delta_1 + 1)(\Delta_2 + 1) \leq n + 1$ then $G_1$ and $G_2$ pack.
Bollobás-Eldridge Graph Packing Conjecture [1978]: If \((\Delta_1 + 1)(\Delta_2 + 1) \leq n + 1\) then \(G_1\) and \(G_2\) pack.

If true, this conjecture would be sharp, and would be a considerable extension of the Hajnal-Szemerédi theorem on equitable colorings.

The conjecture has only been proved when

\(\Delta_1 \leq 2\) [Aigner + Brandt (1993), and Alon + Fischer (1996)], or

\(\Delta_1 = 3\) and \(n\) is huge [Csaba + Shokoufandeh + Szemerédi (2003)].
Let us consider a refinement of the Bollobás-Eldridge Conjecture.

Conjecture: For a fixed $0 \leq \epsilon \leq 1$. If $(\Delta_1 + 1)(\Delta_2 + 1) \leq \frac{n}{2}(1 + \epsilon) + 1$, then $G_1$ and $G_2$ pack.

For $\epsilon = 0$, this is essentially the Sauer-Spencer Theorem, while $\epsilon = 1$ gives the Bollobás-Eldridge conjecture.

For any $\epsilon > 0$ this would improve the Sauer-Spencer result (in a different way than Theorem 1).
Theorem 2 [Kaul + Kostochka + Yu, 2005+]:
For \( \epsilon = 0.2 \), and \( \Delta_1, \Delta_2 \geq 400 \),
If \( (\Delta_1 + 1)(\Delta_2 + 1) \leq \frac{n}{2}(1 + \epsilon) + 1 \), then \( G_1 \) and \( G_2 \) pack.
Theorem 2 [Kaul + Kostochka + Yu, 2005+]:
For $\epsilon = 0.2$, and $\Delta_1, \Delta_2 \geq 400$,
If $(\Delta_1 + 1)(\Delta_2 + 1) \leq \frac{n}{2}(1 + \epsilon) + 1$, then $G_1$ and $G_2$ pack.

In other words,

Theorem 2 [Kaul + Kostochka + Yu, 2005+]:
For $\Delta_1, \Delta_2 \geq 400$,
If $(\Delta_1 + 1)(\Delta_2 + 1) \leq (0.6)n + 1$, then $G_1$ and $G_2$ pack.

This is work in progress.
Some Proof Ideas for Theorem 1

Theorem 1 [Kaul + Kostochka, 2005]:
If \(2\Delta_1\Delta_2 \leq n\), then
\(G_1\) and \(G_2\) do not pack if and only if
one of \(G_1\) and \(G_2\) is a perfect matching and the other
either is \(K_{\frac{n}{2}, \frac{n}{2}}\) with \(\frac{n}{2}\) odd or contains \(K_{\frac{n}{2}+1}\).

We have to analyze the ‘minimal’ graphs that do not
pack (under the condition \(2\Delta_1\Delta_2 \leq n\)).

\((G_1, G_2)\) is a critical pair if \(G_1\) and \(G_2\) do not pack, but for
each \(e_1 \in E(G_1)\), \(G_1 - e_1\) and \(G_2\) pack, and for each
\(e_2 \in E(G_2)\), \(G_1\) and \(G_2 - e_2\) pack.
Each bijection $f : V_1 \rightarrow V_2$ generates a (multi)graph $G_f$, with

$$V(G_f) = \{(u, f(u)) : u \in V_1\}$$

$$(u, f(u)) \leftrightarrow (u', f(u')) \iff uu' \in E_1 \text{ or } f(u)f(u') \in E_2$$
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![Diagram showing the transformation of $G_1$ to $G_f$ through $f$](image)
$(u_1, u_2)$-switch means replace $f$ by $f'$, with

$$f'(u) = \begin{cases} 
  f(u) & u \neq u_1, u_2 \\
  f(u_2) & u = u_1 \\
  f(u_1) & u = u_2 
\end{cases}$$
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\]

2-neighbors of \(u_1 \leftrightarrow 2\)-neighbors of \(u_2\)
Some Proof Ideas for Theorem 1

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\]

\(G_f\) \rightarrow \text{(u_1 , u_2)-switch} \rightarrow G_{f'}
An important structure that we utilize in our proof is -

\((u_1, u_2; 1, 2)-\text{link}\) is a path of length two (in \(G_f\)) from \(u_1\) to \(u_2\) whose first edge is in \(E_1\) and the second edge is in \(E_2\).

For \(e \in E_1\), an \textit{e-packing (quasi-packing)} of \((G_1, G_2)\) is a bijection \(f\) between \(V_1\) and \(V_2\) such that \(e\) is the only edge in \(E_1\) that shares its incident vertices with an edge from \(E_2\).

Such a packing exists for every edge \(e\) in a critical pair.
Outline of the Proof of Theorem 1

The main tool –

Lemma 1: Let \((G_1, G_2)\) be a critical pair and \(2\Delta_1\Delta_2 \leq n\). Given any \(e \in E_1\), in a \(e\)–packing of \((G_1, G_2)\) with \(e = u_1u'_1\), the following statements are true.

(i) For every \(u \neq u'_1\), there exists either a unique \((u_1, u; 1, 2)\)–link or a unique \((u_1, u; 2, 1)\)–link,

(ii) there is no \((u_1, u'_1; 1, 2)\)–link or \((u_1, u'_1; 2, 1)\)–link,

(iii) \(2\deg_{G_1}(u_1)\deg_{G_2}(u_1) = n\).
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The main tool –

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(ii) there is no \((u_1, u'_1; 1, 2)\)-link or \((u_1, u'_1; 2, 1)\)-link,

(iii) \(2\deg_{G_1}(u_1)\deg_{G_2}(u_1) = n\).
Lemma 2: If \(2\Delta_1\Delta_2 = n\) and \((G_1, G_2)\) is a critical pair, then every component of \(G_i\) is either \(K_{\Delta_i, \Delta_i}\) with \(\Delta_i\) odd, or an isolated vertex, or \(K_{\Delta_i+1}, \ i = 1, 2\).

Lemma 2 allows us to settle the case of: \(\Delta_1\) or \(\Delta_2 = 1\).

Then, we have to give a packing for all remaining pairs of graphs, to eliminate their possibility.
The following Lemma limits the possible remaining pairs of graphs.

**Lemma 3**: Let $\Delta_1, \Delta_2 > 1$ and $2\Delta_1\Delta_2 = n$. If $(G_1, G_2)$ is a critical pair and the conflicted edge in a quasi-packing belongs to a component $H$ of $G_2$ isomorphic to $K_{\Delta_2, \Delta_2}$, then every component of $G_1$ sharing vertices with $H$ is $K_{\Delta_1, \Delta_1}$.

Now, we completely eliminate such graphs.

**Lemma 4**: Suppose that $\Delta_1, \Delta_2 \geq 3$ and odd, and $2\Delta_1\Delta_2 = n$. If $G_1$ consists of $\Delta_2$ copies of $K_{\Delta_1, \Delta_1}$ and $G_2$ consists of $\Delta_1$ copies of $K_{\Delta_2, \Delta_2}$, then $G_1$ and $G_2$ pack.
Outline of the Proof of Theorem 1

Now, let's eliminate the only remaining possibility.

**Lemma 5:** Let $\Delta_1, \Delta_2 > 1$ and $2\Delta_1\Delta_2 = n$. If every non-trivial component of $G_i$ is $K_{\Delta_i+1}$, $i = 1, 2$, then $G_1$ and $G_2$ pack.

This would complete the proof of Theorem 1.