Chromatic Number of the Square of Kneser graph $K^2(2k+1, k)$

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Joint work with Jeong Hyun Kang, UWG.

Defn Kneser graph $K(n,k)$

Vertex set $= \binom{[n]}{k} = \{A \subseteq [n] = \{1,2,3,\dots,n\} : |A| = k\}$

Edge set: AB is an edge if $A \cap B = \emptyset$.

Example $K(5,2)$

Vertices are all 2-element subsets of $[5]$

$\{1,2\}$ $\{1,3\}$ $\{1,4\}$ $\{1,5\}$

$\{2,3\}$ $\{2,4\}$ $\{2,5\}$

$\{3,4\}$ $\{3,5\}$

$\{4,5\}$

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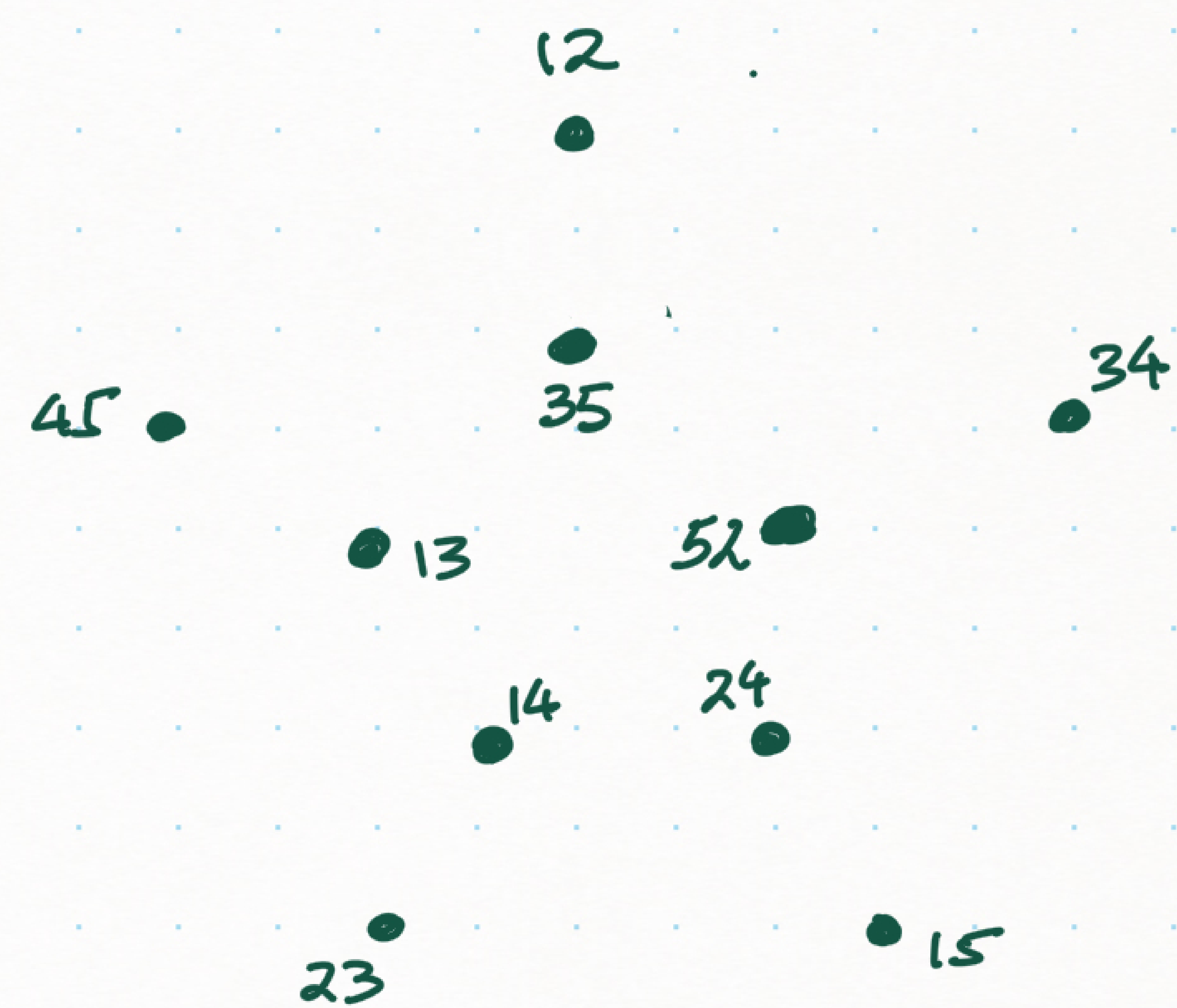
12	13	14	15
23	24	25	
34	25		
45			



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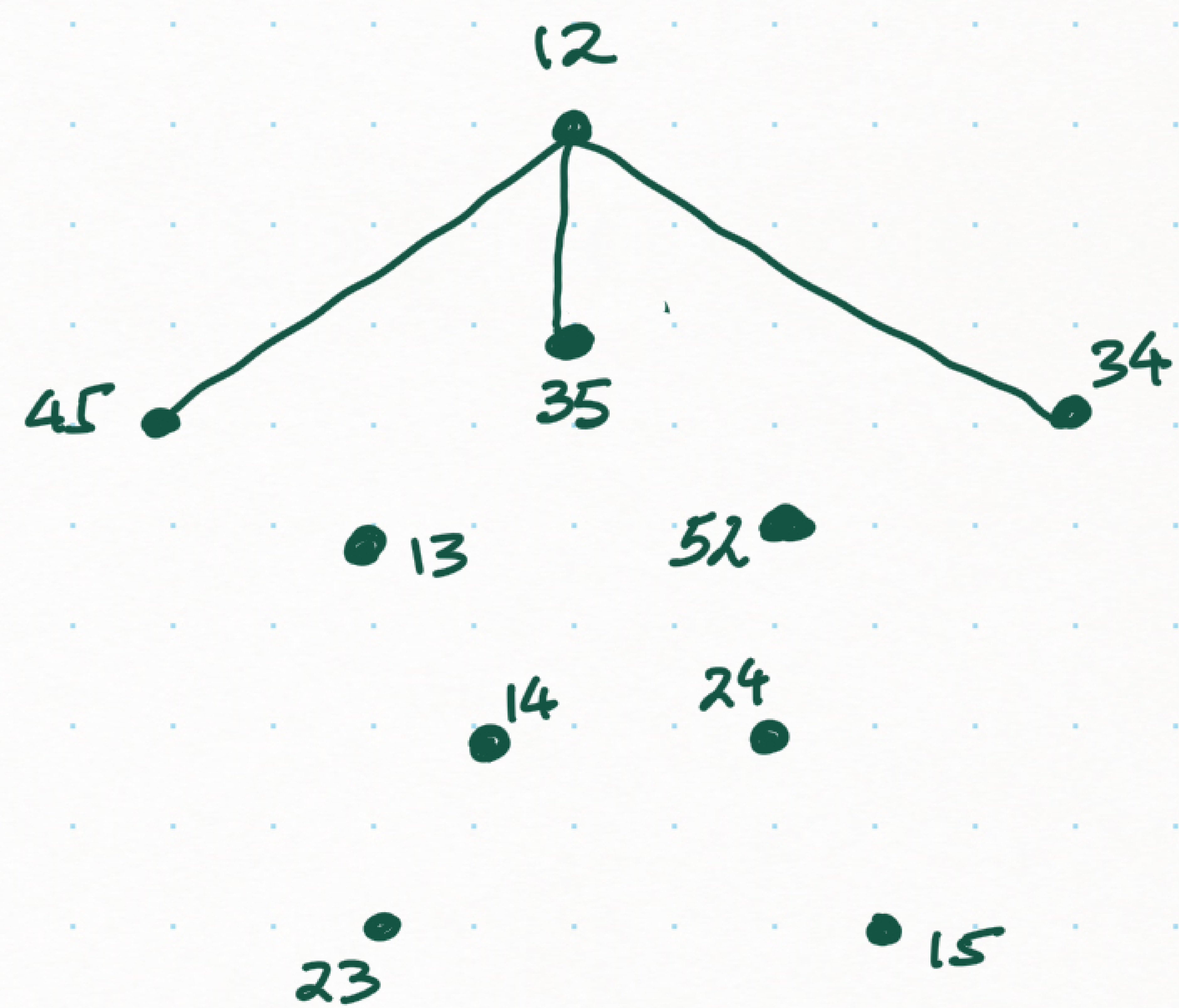

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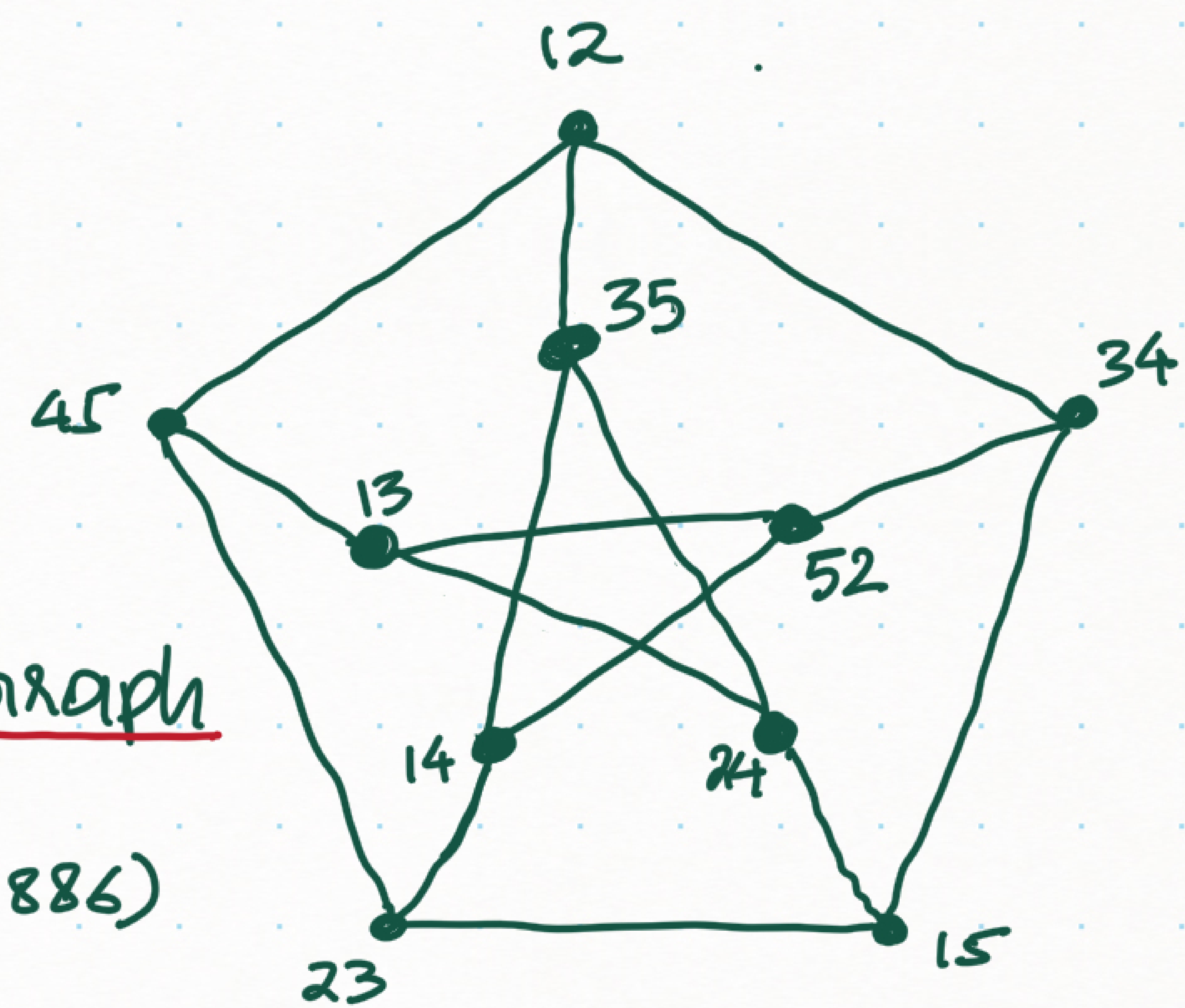
Petersen Graph (1898)

— Kempe (1886)



Examples • $K(n,1) = K_n$

• $K(n,2) = \overline{L(K_n)}$

• $K(2k,k) =$ matching (each $A \subseteq [2k]$ w. $|A| = k$ is adjacent to $\overline{A}$)

• $K(2k+1,k)$ are called $\boxed{\text{Odd Graphs}}$ O_k

↳ Kowaleski (1917)

↳ Biggs (1972)

• Biggs (1979) conjectured O_k is Hamiltonian $\forall k \geq 4$.
Recently proved by Mütze et al. (2018).

• $\chi(K(2k+1,k)) = 3$

• Independent set is an intersecting family of k-sets,
so Erdös-Ko-Rado tells us: $\alpha(K(2k+1,k)) = \binom{2k}{k-1}$

- Kneser (1955) conjectured $\chi(K(n,k)) = n-2k+2$ for all $n \geq 2k$.
(motivated by Kaplansky's work on quadratic forms)

→ Famously proved by Lovasz (1978) using topological methods

- Simplified by Barany (1978) using Borsuk–Ulam Theorem & Gale's Thm.[1]

Birth of Topological methods in Combinatorics[2]

see: ① Aigner, Ziegler, Proofs from the BOOK.
② Matousek, Using Borsuk–Ulam Thm.

- First combinatorial proof by Matousek (2004)

Fundamental Questions about Kneser graphs $K(n,k)$

→ Independence number, $\alpha(K(n,k)) = \binom{n-1}{k-1}$
max size of intersecting family from Erdős-Ko-Rado (1961)

→ Chromatic number, conjectured in 1955 & proved in 1978
partition into intersecting families

→ clique number, $\omega(K(n,k)) = \left\lfloor \frac{n}{k} \right\rfloor$
max size of nonintersecting family from Baranyai's Thm (1975) & it generalization.

→ Fractional chromatic number $\chi^*(K(n,k)) = \chi(K(n,k))$
Johnson-Holyroyd-Stahl conjecture (1997) proved in 2011 (only using Topological methods)

OPEN → Hamiltonicity, conjectured in 1972/79
Only known for $n=2k+1$ (2018) & $n \geq 2.64k$ (2003).

Füredi (2002) asked what is the

$$\chi(K^2(n,k))\ ?^{\circledast}$$

Defn G^2 square of a graph G
has $V(G^2) = V(G)$
and $uv \in E(G^2)$ iff $d_G(u,v) \le 2$

• Proper coloring $f : V(G) \longrightarrow \{1, 2, 3, \ldots, t\}$
with $f(u) \neq f(v)$ if $uv \in E(G)$ i.e., $d_G(u,v) = 1$
or $d_G(u,v) = 2$

⊛ long history of $\chi(G^d)$ in terms of girth & for planar graphs.

$\chi(K^2(n,k))$

Trivial cases

$K(n,k)$ with $n \le 2k-1$ is an independent set

so, $\chi(K(n,k)) = 1$

$K(n,k)$ with $n = 2k$ is a matching

so, $\chi(K(2k,k)) = 2$

$\chi(K^2(n,k))$

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& $\chi(K^2(n,k)) = 1$

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so, $\chi(K(2k,k)) = 2$

& $\chi(K^2(2k,k)) = 2$

$\chi(K^2(n,k))$

Trivial cases

$K(n,k)$ with $n \geq 3k-1$

Note any pair of intersecting k-sets will have a common neighbor in $K(n,k)$ [a k-set disjoint from both] & hence will be adjacent in $K^2(n,k)$.

$K^2(n,k)$ is a clique for $n \geq 3k-1$.

$$\boxed{\chi(K^2(n,k)) = ? \quad \text{for } 2k+1 \leq n \leq 3k-2}$$

We focus on $\underline{K^2(2k+1,k)}$

<u>Chromatic number of the square of the odd graph.</u>

Edge structure of $K^2(2k+1,k)$

- $AB \in E(K(2k+1,k))$ iff $A \cap B = \emptyset$
- $AB \in E(\underbrace{K^2}_{K^2(2k+1,k)} \setminus \underbrace{K}_{K(2k+1,k)})$ iff $A \cap B \neq \emptyset$ & $\exists C$ s.t. $A \cap C = \emptyset$, $B \cap C = \emptyset$

Edge structure of $K^2(2k+1,k)$

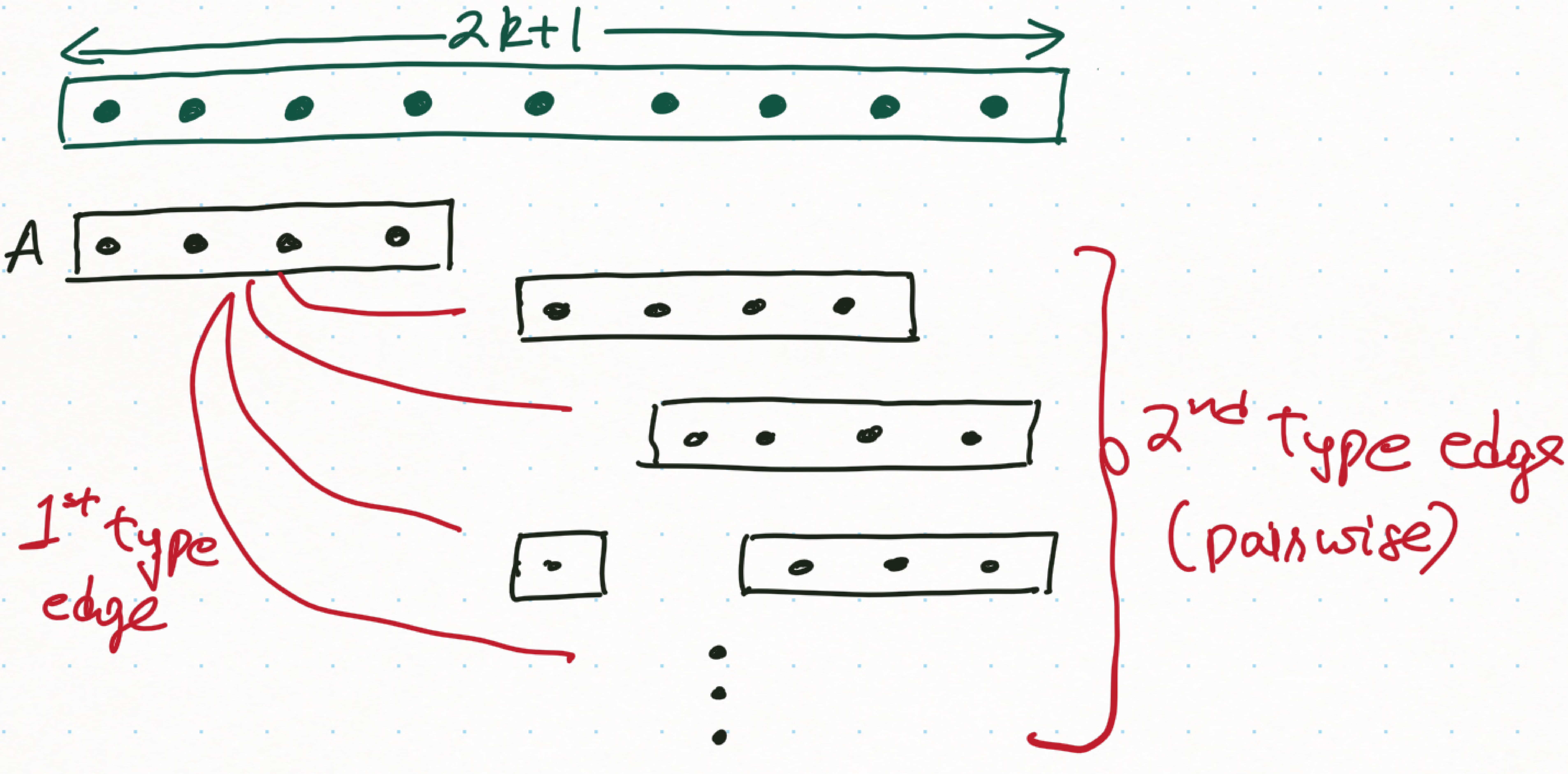
- $AB \in E(K(2k+1,k))$ iff $A \cap B = \emptyset$ "first type"
- $AB \in E(K^2 \setminus K)$ iff $A \cap B \neq \emptyset$ & $\exists C$ s.t. $A \cap C = \emptyset$, $B \cap C = \emptyset$

which can happen

iff $|A \cap B| = k-1$ "second type"

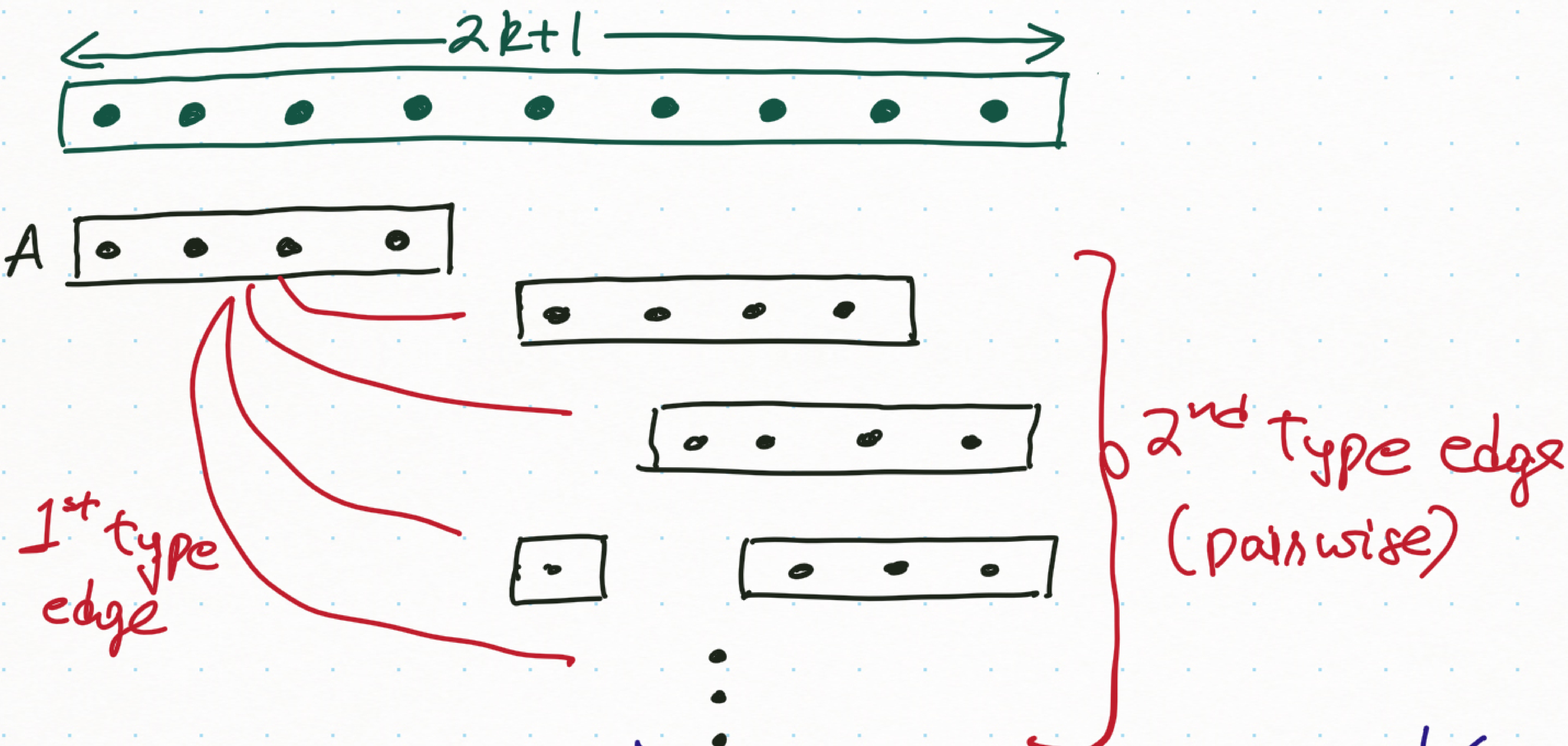
Lower Bound

$$\chi(K^2(2k+1, k)) \geq \omega(K^2)$$



Lower Bound

$$\chi(K^2(2k+1,k)) \geq \underline{\omega(K^2) = k+2}$$



#vertices $1 + \binom{k+1}{k} \longrightarrow K_{k+2}$

Upper bounds

$\boxed{\text{Conjecture } \chi(K^2) \leq 2k + \text{constant for } k \geq 2}$

- Kang 2004 — $4k+3$
- Kim & Nakpsasit 2004 — $4k+2$
- Chen & Lin & Wu 2009 — $3k+2$
- Khodkar & Leach 2009 — $\chi(K^2(9,4)) = 11$
- Kim & Park 2014 — $\frac{8}{3}k + \frac{20}{3}$ for $k \geq 3$
- Kim & Park 2016 — $\frac{32}{15}k + 32$ for $k \geq 7$
- Kang 2018 — $\frac{5}{2}k + 6$ for $k \geq 2$

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$\boxed{\text{Kaul \& Kang 2020} \quad 2(k+1) + 2\lfloor \log_2 k \rfloor}$

Theorem 1 [K. & Kang 2020]

$$\chi(K^2(2k+1,k)) \leq 2(k+1) + 2\lfloor \log_2 k \rfloor$$

Theorem 2 [K. & Kang 2020]

If $k = 2^\alpha - 1$ for some $\alpha \in \mathbb{Z}^+$

then $\chi(K^2(2k+1,k)) \leq 2k+2$

Open Questions

- **Upper Bound** $\chi(K^2(2k+1,k)) \le 2k + \text{constant}$?

  (Remove $+\log_2 k$ factor from our bound)

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- **Upper Bound** $\chi(K^2(2k+1,k)) \le 2k + \text{constant}$?
  (Remove $+\log_2 k$ factor from our bound)

- **Lower Bound** Known: $\chi(K^2) \ge \omega(K^2) = k+2$

Better? $\chi(K^2) \ge \dfrac{|V(K^2)|}{\alpha(K^2)} = \dfrac{\binom{2k+1}{k}}{\alpha(K^2(2k+1,k))}$??

Independent set in $K^2(2k+1,k)$

$= \left\{ A \in \binom{[2k+1]}{k} : 1 \le |A \cap B| \le k-2 \right\}$

An intersecting family of k-sets in $[2k+1]$ with pairwise intersection bounded by $k-2$.

Open Questions

- **Upper Bound** $\chi(K^2(2k+1,k)) \leq 2k + \text{constant}$

  (Remove $+\log_2 k$ factor from our bound)

- **Lower Bound** Known: $\chi(K^2) \geq \omega(K^2) = k+2$

  Better? $\chi(K^2) \geq \dfrac{|V(K^2)|}{\alpha(K^2)} = \dfrac{\binom{2k+1}{k}}{\alpha(K^2(2k+1,k))}$

$$\boxed{\begin{array}{l}\text{Maximize } \left|\left\{A \in \binom{[n]}{k} : d_1 \leq |A \cap B| \leq d_2\right\}\right| \\ \text{where } 0 < d_1 \leq d_1 < k < n\end{array}}$$

upper bound max size of intersecting family with bounded intersection sizes.

Open Questions

- **Upper Bound** $\chi(K^2(2k+1,k)) \leq 2k + \text{constant}$

  (Remove $+\log_2 k$ factor from our bound)

- **Lower Bound** Known: $\chi(K^2) \geq \omega(K^2) = k+2$

  Better? $\chi(K^2) \longrightarrow$ partition into independent sets

How to partition $\binom{[2k+1]}{k}$ into intersecting families with bounded intersection sizes?

Ind. sets are $\{A \in \binom{[2k+1]}{k} : 1 \leq |A \cap B| \leq k-2\}$

Theorem 1 [K. & Kang 2020]

$$\chi(K^2(2k+1,k)) \leq 2(k+1) + 2\lfloor \log_2 k \rfloor$$

Theorem 2 [K. & Kang 2020]

If $k = 2^\alpha - 1$ for some $\alpha \in \mathbb{Z}^+$

then $$\chi(K^2(2k+1,k)) \leq 2k+2$$

Outline of the Proof

Notation

$\binom{X}{s} \equiv$ collection of all subsets of X of size s

Direct sum $\mathcal{A} \oplus \mathcal{B} = \{A \cup B : A \in \mathcal{A} \text{ and } B \in \mathcal{B}\}$
where $\mathcal{A}$ & $\mathcal{B}$ are disjoint families of subsets of $[n]$.

collection of all pairwise unions of sets from $\mathcal{A}$ and $\mathcal{B}$ respectively.

Generalized Johnson Graphs $J(X, \{s, s-1, \ldots, s-\ell\})$
Vertices are all subsets of X of size $s, s-1, \ldots, s-\ell$. Edges when ① $|A| = |B|$ & $|A| = |B| = |A \cap B| + 1$
② $|A| < |B| \Rightarrow A \subseteq B$

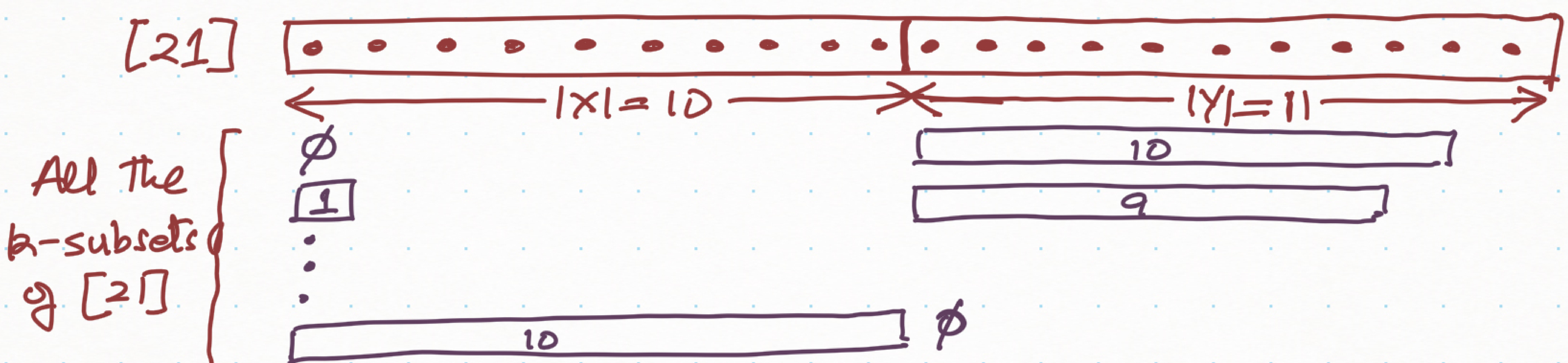
Cartesian Product of Graphs $G \square H$

- **Partition $[2k+1]$** into $X \cup Y$ with $|X| = k$, $|Y| = k+1$

- **Vertex set** $V = \binom{[2k+1]}{k}$ is the disjoint union of direct sums of the form

$$\binom{X}{s} \oplus \binom{Y}{k-s} \quad \text{for } 0 \le s \le k$$

Think of $K^2(21, 10)$ i.e., $k = 10$



Kang (2018) Pair up subsets of the form

$$\binom{X}{\lfloor k/2 \rfloor - i} \oplus \binom{Y}{\lceil k/2 \rceil + i} \quad \text{with} \quad \binom{X}{\lfloor k/2 \rfloor + i} \oplus \binom{Y}{\lceil k/2 \rceil - i}$$

In $K^2(21,10)$ we pair up subsets of the form

Layers

$$\emptyset \oplus \boxed{10} \quad \text{with} \quad \boxed{10} \oplus \emptyset$$

$$\boxed{1} \oplus \boxed{9} \quad \text{with} \quad \boxed{9} \oplus \boxed{1}$$

$$\vdots \qquad\qquad \vdots$$

$$\boxed{4} \oplus \boxed{6} \quad \text{with} \quad \boxed{6} \oplus \boxed{4}$$

$$\boxed{5} \oplus \boxed{5} \quad \text{by itself.}$$

Kang (2018) No edges between layers that are at least two apart.

Kang (2018) Partition V into V_i, $0 \le i \le \lfloor k/2 \rfloor$

where

Layers

$$V_0 = \binom{X}{\lfloor k/2 \rfloor} \oplus \binom{Y}{\lceil k/2 \rceil}$$

$$V_i = \binom{X}{\lfloor k/2 \rfloor - i} \oplus \binom{Y}{\lceil k/2 \rceil - i} \quad \text{for } 1 \le i \le \lfloor k/2 \rfloor$$

when k odd, $V_{\lceil k/2 \rceil} = \binom{X}{k} \oplus \binom{Y}{0}$

$\boxed{\text{For } i,j \text{ with } |i-j| \ge 2, \text{ there are no edges of } K^2(2k+1,k) \text{ between } V_i \text{ and } V_j.}$

So, $K^2[V_{2j+1}]$, $j \ge 0$ can share same set of colors

and $K^2[V_{2j}]$, $j \ge 0$ can share same set of colors.

Lemma 1 For $k \geq 1$,

$$\chi(K^2(2k+1,k)) \leq 2\left\lceil \frac{k+1}{2} \right\rceil + p(k)$$

where
$$p(k) = \max\left\{ 2\left\lceil \frac{k+1}{2} \right\rceil, \chi\left(K^2\left[\underbrace{\binom{X}{\lfloor k/2 \rfloor} \oplus \binom{Y}{\lceil k/2 \rceil}}_{\text{middle layers}}\right]\right)\right\}$$

Idea for non-middle layers

Isomorphism between

$K^2[V_s]$ and $\underbrace{J(X,\{s\}) \,\square\, J(Y,\{s+1,s\})}_{\text{Easy to color!}}$

for $1 \leq s \leq \lfloor \frac{k}{2} \rfloor$

Cartesian Product

Generalized Johnson graphs

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Idea

Isomorphism between

$K^2[V_s]$ and $J(X,\{s\}) \square J(Y,\{s+1,s\})$

when k even.

$K^2[V_s]$ and $J(X,\{s,s-1\}) \square J(Y,\{s\})$

when k odd

and $K^2[V_0]$ and $K^2[V_{\lceil \frac{k}{2} \rceil}]$

can share colors.

<u>Lemma 2</u> For $k \geq 4$,

$$\chi\left(K^2\left[\binom{X}{\lfloor k/2 \rfloor} \oplus \binom{Y}{\lceil k/2 \rceil}\right]\right) \leq \begin{cases} \chi\left(K^2\left(k, \frac{k-1}{2}\right)\right) & \text{if } k \text{ odd} \\ \max\left\{\chi\left(K^2\left(k-1, \frac{k}{2}-1\right)\right), \chi\left(K^2\left(k+1, \frac{k}{2}\right)\right)\right\} & \text{if } k \text{ even} \end{cases}$$

<u>Idea:</u> Give an explicit coloring using the optimal coloring of $K^2(k, \frac{k-1}{2})$ based on the structure of the projections of the set onto X and Y.

More complicated when k even as we have to use optimal colorings of both $K^2(k-1, \frac{k}{2}-1)$ and $K^2(k+1, \frac{k}{2})$

Lemma 2 For $k \geqslant 4$,

$$\chi\left(K^2\left[\binom{X}{\lfloor k/2 \rfloor} \oplus \binom{Y}{\lceil k/2 \rceil}\right]\right) \leq \begin{cases} \chi\left(K^2\left(k, \frac{k-1}{2}\right)\right) & \text{if } k \text{ odd} \\ \max\left\{\chi\left(K^2\left(k-1, \frac{k}{2}-1\right)\right), \chi\left(K^2\left(k+1, \frac{k}{2}\right)\right)\right\} & \text{if } k \text{ even} \end{cases}$$

For k even

$$f(A_X \oplus A_Y) = \begin{cases} g_1(A_X \setminus \{k\}) + g_2(A_Y) \pmod{p} & \text{if } k \in A_X \\ g_1(\overline{A_X} \setminus \{k\}) + g_2(A_Y) \pmod{p} & \text{if } k \notin A_X \end{cases}$$

A k-subset

A_X projection onto X

A_Y projection onto Y

g_1 optimal p_1-coloring of $K^2\left(k-1, \frac{k}{2}-1\right)$

g_2 optimal p_2-coloring of $K^2\left(k+1, \frac{k}{2}\right)$

$p = \max\{p_1, p_2\}$

The main proof carefully applies Lemmas 1 & 2 inductively (via k).

previously known

The bounds are tight when $k=3,4,5,6$ & would stay tight giving a bound of $2k+2$ if we had as nice a bound for k even as we have for k odd.

Since $k=2^{\alpha}-1$ gives all of $k_1=\frac{k-1}{2}$, $k_2=\frac{k_1-1}{2}$, $\ldots\ldots$, are odd.

We only need the odd portion of Lemma 2 there.

Theorem 1 [K. & Kang 2020]

$$\chi(K^2(2k+1,k)) \leq 2(k+1) + 2\lfloor \log_2 k \rfloor$$

Theorem 2 [K. & Kang 2020]

If $k = 2^\alpha - 1$ for some $\alpha \in \mathbb{Z}^+$

then $\chi(K^2(2k+1,k)) \leq 2k+2$

$\boxed{\text{Open Questions}}$ ① Remove the $\log_2 k$ factor above

② Ideas for $K^2(2k+r,k)$ with $2 \leq r \leq k-2$?

③ Lower bound for $\chi(K^2(n,k))$

Partition $\binom{[n]}{k}$ into intersecting k-uniform families with bounded intersection sizes.

Thank You!!