The Art Gallery Problem

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The original art gallery problem (V. Klee, 1973) asked for the minimum number of guards sufficient to see every point of the interior of an $n$-vertex simple polygon.

A simple polygon is a simply-connected closed region whose boundary consists of a finite set of line segments.

Two points $u$ and $v$ in a polygon are said to be mutually visible if the line segment joining $u$ and $v$ lies inside the polygon.
How many guards are needed
How many guards are needed
Chvátal (1975) proved that $\left\lfloor \frac{n}{3} \right\rfloor$ guards are always sufficient for any $n$ vertex simple polygon.

Fisk gave a simple proof using graph coloring in 1978.
Proof of Art Gallery Theorem
Step 1: Triangulation
Step 2: Proper coloring using 3 colors.

Proper coloring of the vertices means adjacent vertices get different colors.
Proof of Art Gallery Theorem

Step 3: Choose minimum size color class
Where to place the guards?

Note that Fisk’s proof gives an explicit placement of guards on the vertices (corners) of the polygon.

**Vertex guard** means the guard should be positioned on a vertex.

Other **variations** on guards include:

**Point guard** means the guard can be placed anywhere in the polygon.

**Edge guard** means the guard can be placed anywhere along an edge of the polygon.

**Mobile guard** means the guard is allowed to patrol along a line segment lying in the polygon.
What does an Art Gallery look like?

Walls tend to be perpendicular to each other!

An **orthogonal art gallery** is a polygon whose edges are all aligned with a pair of orthogonal coordinate axes, which we take to be horizontal and vertical. Internal angles are either $\frac{\pi}{2}$ or $\frac{3\pi}{2}$.
What does an Art Gallery look like?

There are usually obstructions to visibility in the interior.

Polygons with holes
Guards can not see through holes.
O’Rourke 1987: \( \lceil \frac{n+2h}{3} \rceil \) vertex guards are sufficient for a polygon with \( n \) vertices and \( h \) holes.

Shermer’s conjecture I, 1982

Any polygon with \( n \) vertices and \( h \) holes can be guarded by \( \lceil \frac{n+h}{3} \rceil \) vertex guards.

Hoffmann, Kaufmann, Kriegel 1991: \( \lceil \frac{n+h}{3} \rceil \) point guards are sufficient for a polygon with \( n \) vertices and \( h \) holes.

This is sharp.
Kahn, Klawe and Kleitman (1983) proved that \( \left\lfloor \frac{n}{4} \right\rfloor \) vertex guards are sufficient for a \( n \)-vertex orthogonal art gallery.
Proof of Orthogonal Art Gallery Theorem
Proof of Orthogonal Art Gallery Theorem

Step 1: Convex Quadrilateralization
Proof of Orthogonal Art Gallery Theorem

Step 2a: Add diagonals
Proof of Orthogonal Art Gallery Theorem

Step 2b: Proper Coloring using 4 colors.
Proof of Orthogonal Art Gallery Theorem

Step 3: Choose minimum size color class
A Short Survey II Orthogonal Polygons w. holes

O’Rourke 1987: \( \left\lfloor \frac{n+2h}{4} \right\rfloor \) vertex guards are sufficient for an orthogonal polygon with \( n \) vertices and \( h \) holes.

Shermer’s conjecture II, 1982
Any orthogonal polygon with \( n \) vertices and \( h \) holes can be guarded by \( \left\lfloor \frac{n+h}{4} \right\rfloor \) vertex guards.

Hoffmann, Kriegel 1996: \( \left\lfloor \frac{n}{3} \right\rfloor \) vertex guards are sufficient for an orthogonal polygon with \( n \) vertices and \( h \) holes.

Hoffmann 1990: \( \left\lfloor \frac{n}{4} \right\rfloor \) point guards are sufficient for a polygon with \( n \) vertices and \( h \) holes.

These are not sharp.
In 1981, Godfried Toussaint formulated the problem for edge and mobile guards.

**O’Rourke 1983**: $\left\lfloor \frac{n}{4} \right\rfloor$ mobile guards are sufficient for a simple polygon with $n$ vertices.

This is sharp.

**Toussaint’s conjecture, 1981**

Except for a few polygons, every simple polygon with $n$ vertices can be guarded by $\left\lfloor \frac{n}{4} \right\rfloor$ edge guards.
A Short Survey III contd.  Edge and Mobile Guards

Aggarwal 1984: \(\left\lfloor \frac{3n+4}{16} \right\rfloor\) mobile or edge guards are sufficient for a simple orthogonal polygon with \(n\) vertices.

Improved to just edge guards by Bjorling-Sachs, 1998

Gyori et al. 1996: \(\left\lfloor \frac{3n+4h+4}{16} \right\rfloor\) mobile guards are sufficient for an orthogonal polygon with \(n\) vertices and \(h\) holes.

These results are sharp.
A Short Survey IV

Other variations:

- **Art galleries with interior walls** (diagonal joining two corners with small doorway placed arbitrarily).
- **Half-guard** (fixed $180^\circ$ field of vision): $\frac{n}{3}$ point half-guards.
- **Guarded guards** (each guard is also visible to at least one other guard): $\frac{3n-1}{7}$ vertex guarded-guards; $\frac{n}{3}$ vertex guarded-guards for orthogonal polygons.
- **3-D polytopes**.
- **Fortress Problem**: Guards on edges so that exterior is visible.
- **Prison yard Problem**: Guards on edges so that both interior and exterior are visible.
Orthogonal Art Gallery Problem

Kahn, Klawe and Kleitman 1983: $\left\lfloor \frac{n}{4} \right\rfloor$ vertex guards are sufficient for a $n$-vertex orthogonal art gallery.

What about Orthogonal Art galleries with holes?
How many guards are needed for guarding an orthogonal art gallery with $n$ vertices and $h$ holes?

**Shermer’s conjecture, 1982**

Any orthogonal polygon with $n$ vertices and $h$ holes can be guarded by $\left\lfloor \frac{n+h}{4} \right\rfloor$ vertex guards.
### Known results

Shermer’s conjecture for $h = 0$

**Kahn, Klawe and Kleitman 1983:** $\left\lceil \frac{n}{4} \right\rceil$ vertex guards are sufficient for a $n$-vertex orthogonal art gallery.

Shermer’s conjecture for $h = 1$

**Agarwal 1984:** $\left\lceil \frac{n+1}{4} \right\rceil$ vertex guards are sufficient for an orthogonal polygon with $n$-vertices and 1-hole.

**O’Rourke 1987:** $\left\lceil \frac{n+2h}{4} \right\rceil$ vertex guards are sufficient for an orthogonal polygon with $n$ vertices and $h$ holes.

**P. Zylinski 2006:** $\left\lceil \frac{n+h}{4} \right\rceil$ vertex guards are sufficient if the dual graph of orthogonal polygon is a **cactus**.

A **cactus** is a graph whose any two cycles share at most one vertex and any such vertex is a cut-vertex.
Y. Jo, K. 2009+: $\left\lfloor \frac{n+\frac{5}{3}h}{4} \right\rfloor$ vertex guards suffice for an orthogonal gallery with $n$ vertices with $h$ holes.

This result improves the O’Rourke bound of $\left\lfloor \frac{n+2h}{4} \right\rfloor$.

O’Rourke uses 6 extra guards per 12 holes.

We use 5 extra guards per 12 holes.

While Shermer’s conjecture wants to use only 3 extra guards per 12 holes.
Outline of our proof
Outline of our proof

- Step 1: Convex Quadrilateralization
Outline of our proof
Outline of our proof

Dual Graph
Outline of our proof

- **Step 2**: Removal of all the exterior leaves in the dual graph.

  A **leaf** is a vertex with only one neighbor.
Outline of our proof
Outline of our proof

Cycle_edges: 
Cycle_vertices:
Path_edges: 
Path_vertices:
Outline of our proof

- **Step 3**: Cut around a boundary vertex of type 3C (or 3C1P or 4C) which has a neighbor of the type 2C or 2C1P or 2C2P or 2C2C.
  By a lemma, such a vertex exists on the boundary of every block with more than one cycle in the dual graph.

**Cutting** means duplicating the vertices of the cut quadrilateral.
Outline of our proof
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We are left with an art gallery whose dual graph is a cactus.
Outline of our proof
Outline of our proof

- **Step 4**: Apply the P. Zylinski result to color the remaining polygon.

  By a lemma, a polygon with $w$ holes whose corresponding dual graph is a **cactus** is $4$-colorable using at most $w$ duplicate vertices.

  Duplicating a vertex allows us to place two guards in the same quadrilateral if needed.
Outline of our proof
Outline of our proof
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- **Step 5a**: Extend the coloring to all of the polygon except quadrilaterals that were cut in step 3 by reattaching the leaves of the dual graph removed in Step 2.
Outline of our proof
Outline of our proof

- **Step 5b:**
  - Attach the cut quadrilaterals to the main colored polygon in reverse order of their being cut (by application of Step 3).
  - At least 3 duplicated vertices of a cut quadrilateral are attached. (Other vertices remain duplicated.)
  - Recoloring of some vertices may be needed to maintain a proper coloring of the extended polygon.
Outline of our proof
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Outline of our proof
Outline of our proof
Let $k$ vertices of type $3C$ be cut.
Let $l$ vertices of type $3C1P$ be cut.
Let $m$ vertices of type $4C$ be cut in Step 3.
Let $w$ be the number of holes remaining when the dual graph becomes a cactus.

Number of duplicate vertices used for the 4-coloring:

$w + (6k - 3k) + (6l - 3l) + (8m - 3m) = w + 3k + 3l + 5m$

Total number of holes in original polygon:

$h = w + 2k + 2l + 3m$

So, the number of duplicate vertices is

$w + 3k + 3l + 5m \leq \frac{5}{3} h.$

Hence, $\left\lfloor \frac{n + \frac{5}{3} h}{4} \right\rfloor$ is the maximum size of the smallest color class.