Some Open Problems

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Delay Coloring

Conjecture 1 [Wilfong, Haxell, and Winkler, 2001]

Let G be a bipartite multigraph with partition classes A, B and maximum degree d. Suppose that each edge e has an associated integer 'delay' $r(e)$.
Then G admits an edge $d+1$-coloring $f : E(G) \to \{0, \dots, d\}$ such that f is proper on A and $f + r (\bmod d+1)$ is proper on B.

When the graph consists of just two vertices joined by d parallel edges, this is implied by a theorem of Marshall Hall (1952):

Given an Abelian group of order m and an multiset of m elements $\{r_e\}$ that sum to identity, there is permutation π for which $\pi_e + r_e$ are all distinct.

(so add a dummy edge with r_e chosen to make the sum equal to 0.)

Alon and Asodi, 2007 proved it asymptotically (in terms of d) for simple bipartite graphs.

Tools: Semi-random coloring procedure.

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Conjecture 1 is the special case of Conjecture 2 with “addition” as “permutation”.

This strengthens the Brualdi-Stein conjecture that every Latin square of order $n \times n$ has a transversal of size $n-1$.

Georgkapoulos 2013+ proved it for $d = 3$.
Explicit coloring after decomposing G into a 2-factor $\mathcal{C}$ and a matching M:

For every $C \in \mathcal{C}$, there is a (greedy) 4-coloring of $M_{C \cap A}$ such that for every 4-coloring of $M_{C \cap B}$, there is a 4-coloring of $E(C)$ such that these give the required coloring.

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Delay Coloring

Question: What about using a few extra colors?

e.g. Alon and Asodi used $o(d)$ extra colors.

Question: What about “approximate” colorings?

Question: For special families of graphs (with bounded degrees)?

Finding a Large Bipartite Subgraph

In a given graph G, find a bipartition (cut) (X, Y), with $X \subseteq V(G)$ and $Y = V(G) \setminus X$, that maximizes the number of edges between X and Y.

$b(G)$ be the number of edges in a largest bipartite subgraph of G.

- Extremal results like Edwards-Erdős Inequalities :
  1) $b(G) \geq \frac{1}{2}m + \frac{1}{8}(\sqrt{8m+1} - 1)$, $m = |E(G)|$
  2) $b(G) \geq \frac{1}{2}m + \frac{1}{4}(n-1)$, $n = |V(G)|$

A local search algorithm

Idea : Starting with an arbitrary vertex partition, switch a vertex from one partite set to the other if doing so increases the number of edges in the cut (the bipartite subgraph induced by the vertex partition).

Given a partition $V(G) = X \cup Y$ of the vertex set of a graph G, a local switch moves a vertex v from X to Y that has more neighbors in X than in Y.

A list of local switches performed successively is a switching sequence.

Size of the bipartite subgraph : How big a bipartite subgraph is guaranteed at the end of a switching sequence?

Length of a switching sequence : How long can a switching sequence be?

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Size of the Bipartite Subgraph

Erdős: Each local switch increases the number of edges in the cut, so the algorithm has to stop. When the algorithm stops, at least half the edges incident to each vertex are in the cut, so the final bipartite subgraph contains at least half the edges of G.

Bylka + Idzik + Tuza, 1999: A bipartite subgraph of size $\frac{1}{2}m + \frac{1}{4}o(G)$ is guaranteed, where $o(G)$ is the number of odd degree vertices in G.

A slight modification of the local switching rules improves the guarantee to the first Edwards-Erdős Inequality : $b(G) \geq \frac{1}{2}m + \frac{1}{8}(\sqrt{8m+1} - 1)$.

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Minimum length of a switching length

Let $s(G)$ denote the minimum length of a maximal switching sequence starting from the trivial vertex partition.

Theorem [Kaul & West, 2008]: If G is an n-vertex loopless multigraph, then $s(G) \leq n/2$.
In fact, there exists a sequence of at most $n/2$ switches that produces a globally optimal partition.

Maximum length of a switching length

Observation: The maximum length of a switching sequence, $l(G)$, is at most $b(G) \leq e(G)$.

This is best possible, as the star $K_{1,n-1}$ achieves equality for both $b(G)$ and $e(G)$.

To get a better upper bound, we look at the tradeoff between $\delta(G)$, the minimum degree of G, and $b(G)$, as a switching sequence progresses.

Theorem [Kaul & West, 2008]: The length of any switching sequence is at most $b(G) - (\frac{3}{8}\delta^2(G) + \delta(G))$.

Let G be triangle-free, then the upper bound above improves to $b(G) - \frac{7}{16}\delta^2(G)$.

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Maximum length of a switching length

Bounding the length with n : A bipartite graph on n vertices has at most $\frac{n^2}{4}$ edges, so any switching sequence has length at most $\frac{n^2}{4}$.
Can we do faster than $\frac{1}{4}n^2$ switches to reach a local optima?

Cowen & West, 2002: When n is a perfect square, there exists a graph G with n vertices that has a switching sequence of length $e(G) = \frac{1}{2}n^{\frac{3}{2}}$.

This gave hope that $l(G) \leq O(n^{\frac{3}{2}})$.

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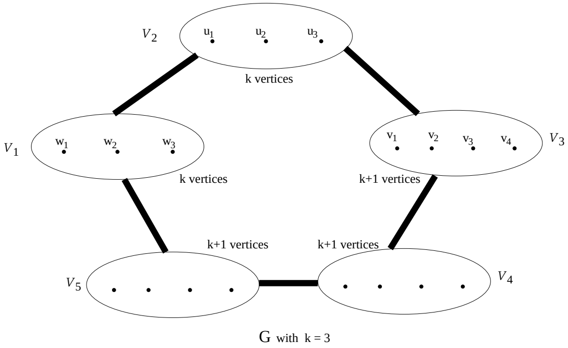
Maximum length of a switching length

Theorem [Kaul & West, 2008]: For every n, there exists a graph G with n vertices that has a switching sequence of length at least $\frac{2}{25}(n^2 + n - 31)$.



Open Questions

Problem 1. Determine the exact constant multiple (between $\frac{8}{100}$ and $\frac{25}{100}$) of n^2 for $I(G)$.

Problem 2. New ideas for upper bounds on $I(G)$.

Open Questions

Modify the switching algorithm by allowing up to $k \geq 1$ vertices to be switched at a time.

How close can we get to the second Edwards-Erdős Inequality: $b(G) \geq \frac{1}{2}m + \frac{1}{4}(n-1)$?

Problem 3. [Tuza, 2001] Given k, determine the largest constant $c = c(k)$ such that the local switching algorithm guarantees a bipartite subgraph of size at least $\frac{1}{2}m + cn - o(n)$.

A construction shows that $c(k) < \frac{1}{4}$, for all k.

What is the smallest k with $c(k) > 0$? Is $c(1) > 0$?

Graph Packing

Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be graphs of order at most n.

G_1 and G_2 are said to *pack* if there exist injective mappings of the vertex sets into $[n]$,
$V_i \to [n] = \{1, 2, \ldots, n\}, i = 1, 2,$
such that the images of the edge sets do not intersect.

- there exists a bijection $V_1 \leftrightarrow V_2$ such that $e \in E_1 \Rightarrow e \notin E_2$.
- G_1 is a subgraph of $\overline{G_2}$.

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We may assume $|V_1| = |V_2| = n$ by adding isolated vertices.

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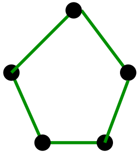
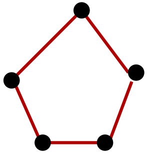


C_5



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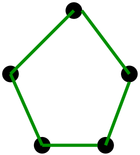

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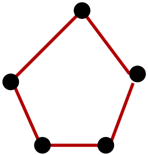

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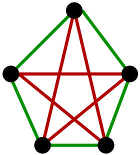

Packing

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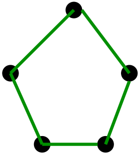

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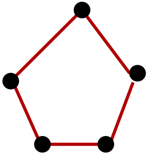

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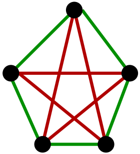

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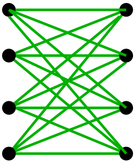
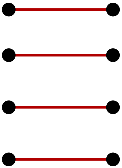


$K_{4,4}$



$4\ K_2$

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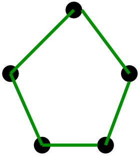

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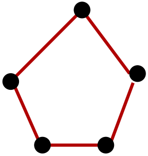

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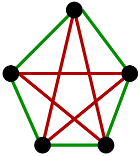

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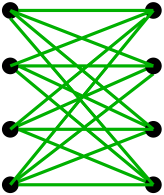

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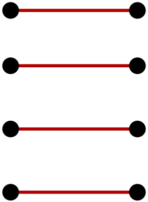

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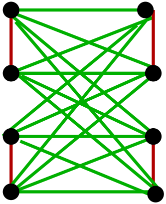

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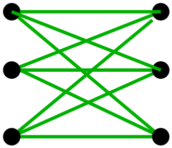
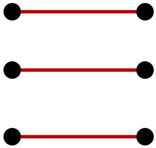


$K_{3,3}$



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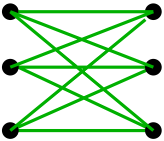

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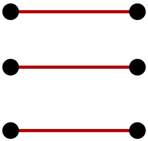

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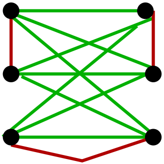

No Packing

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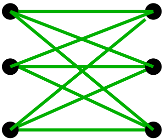

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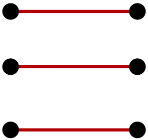

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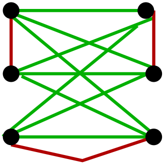

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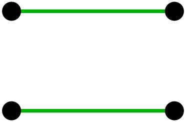
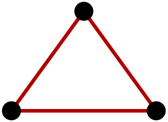
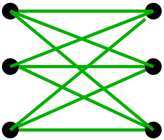


$2\ K_2$



K_3

Examples and Non-Examples



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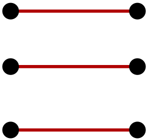

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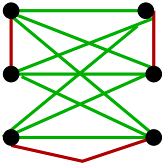

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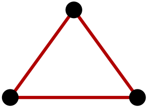

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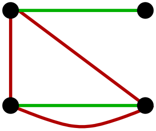

No Packing

Packing Graphs

Existence of a subgraph H in G : Whether H packs with $\overline{G}$.

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Existence of a subgraph H in G : Whether H packs with $\overline{G}$.

"Many" problems in Extremal Graph Theory can be interpreted as a Graph Packing problem.

- Hamiltonian Cycle in graph G : Whether the n-cycle C_n packs with $\overline{G}$.
- Equitable k-coloring of graph G : (A proper k-coloring of G such that sizes of all color classes differ by at most 1) Whether G packs with k cliques of order n/k.
- Turán-type problems : Every graph with more than $ex(n, H)$ edges must pack with $\overline{H}$.
- Ramsey-type problems.

Packing Graphs

Some examples:

Theorem: If $e(G_1)e(G_2) < \binom{n}{2}$, then G_1 and G_2 pack.

Proof. HW for students.

Sharp for star and matching.

Theorem [Bollobas + Eldridge, 1978, & Teo + Yap, 1990]: If Δ_1, $\Delta_2 < n - 1$, and $e(G_1) + e(G_2) \leq 2n - 2$, then G_1 and G_2 do not pack if and only if they are one of the thirteen specified pairs of graphs.

Theorem [Sauer + Spencer, 1978] :
If $2\Delta_1\Delta_2 < n$, then G_1 and G_2 pack.

Kaul and Kostochka, 2007, characterized sharpness as: G_1 and G_2 is a perfect matching and the other either is $K_{\frac{n}{2},\frac{n}{2}}$ with $\frac{n}{2}$ odd or contains $K_{\frac{n}{2}+1}$.

Packing Graphs

Some Conjectures:

Erdős-Sos Conjecture (1963) : Let G be a graph of order n and T be a tree of size k. If $e(G) < \frac{1}{2}n(n-k)$ then T and G pack.

Known only for special classes of trees, etc.

Tree Packing Conjecture (Gyarfas $\sim$ 1968) : Any family of trees $T_2, \ldots, T_n$, where T_i has order i, can be packed.

Known for special classes of trees, and for a sequence of $n/\sqrt{2}$ such trees (Bollobas, 1983).

Bollobás-Eldridge Graph Packing Conjecture [1978] :
If $(\Delta_1+1)(\Delta_2+1) \leq n+1$ then G_1 and G_2 pack.

Kaul, Kostochka, Yu, 2008, proved $(\Delta_1+1)(\Delta_2+1) \leq (0.6)n+1$ suffices.

Packing Families of Graphs

Let $\mathcal{G}_1$ and $\mathcal{G}_2$ be families of graphs of order n, then $\mathcal{G}_1$ and $\mathcal{G}_2$ pack if there exists $G_1 \in \mathcal{G}_1$ and $G_2 \in \mathcal{G}_2$ such that G_1 and G_2 pack.

Note: A family $\mathcal{G}$ and its dual (the family of graphs whose complements are not in $\mathcal{G}$) cannot pack.

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The major application of graph packing results has been to proving lower bounds on computational complexity of graph properties (depth of the decision trees).

Friedgut, Kahn and Widgerson (2003) argue (and give conjectures) that results on packing of families of graphs are needed for improving such complexity bounds.

Restrictive Packing of Graph Families

Let $\mathcal{G}_1$ and $\mathcal{G}_2$ be families of labeled graphs with vertex sets all labeled as $\{v_1, \ldots, v_n\}$.

We want to find $G_1 \in \mathcal{G}_1$ and $G_2 \in \mathcal{G}_2$ such that the identity bijection between $V(G_1)$ and $V(G_2)$ gives a packing of G_1 and G_2.

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We disallow this now.

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We disallow this now.

In particular, we are interested in families of graphs defined in terms of realizations of fixed degree sequences.

Degree Sequence Packing

Let $\pi_1 = (d_1^{(1)}, \ldots, d_n^{(1)})$ and $\pi_2 = (d_1^{(2)}, \ldots, d_n^{(2)})$ be graphic sequences

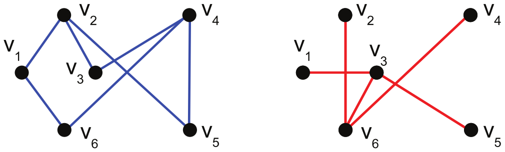
π_1 and π_2 pack if there exist $G_1 = G(\pi_1)$ and $G_2 = G(\pi_2)$ with

$$V(G_1) = V(G_2) = \{v_1, \ldots, v_n\},$$

$$E(G_1) \cap E(G_2) = \emptyset,$$

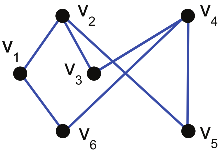
$$\deg_{G_1}(v_i) = d_i^{(1)} \text{ and } \deg_{G_2}(v_i) = d_i^{(2)}.$$

An Example



(2,3,2,2,3,2) (1,1,3,3,1,1)

An Example



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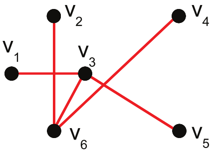

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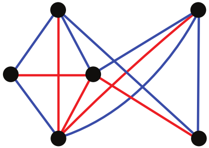

(3,4,5,5,4,3)

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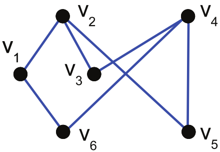

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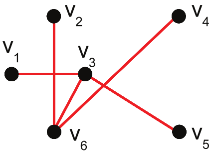

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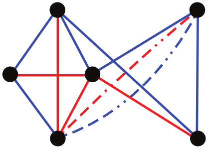

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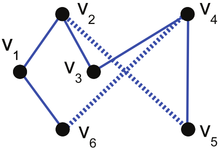

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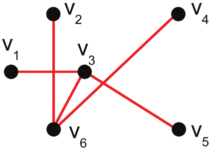

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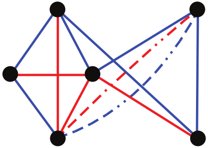

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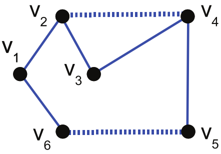

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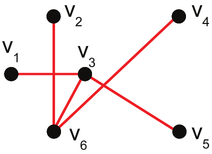

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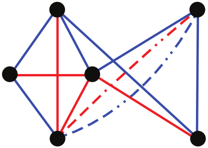

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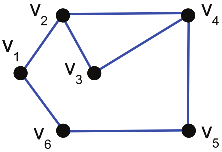

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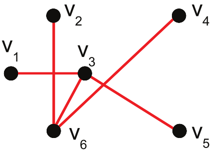

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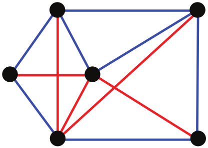

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The Conjectures

Conjecture

Let $\pi_1 = (d_1^{(1)}, \ldots, d_n^{(1)})$ and $\pi_2 = (d_1^{(2)}, \ldots, d_n^{(2)})$ be graphic sequences with δ the least entry in $\pi_1 + \pi_2$.

If $\delta \geq 1$ and $\max\{d_i^{(1)} d_i^{(2)}\} < n/2$, then π_1 and π_2 pack.

Conjecture

Let n be even and let $\pi = (d_1, \ldots, d_n)$ be a graphic seq such that $\pi - k = (d_1 - k, \ldots, d_n - k)$ is graphic for some $k > 0$.

Then there exists a realization G of π that contains k edge-disjoint 1-factors.

A Graph Packing Result

Theorem (Sauer–Spencer, 1978)

Let G_1 and G_2 be n-vertex graphs with max degrees Δ_1 and Δ_2.

If $\Delta_1\Delta_2 < n/2$, then G_1 and G_2 pack.

A Graph Packing Result

Theorem (Busch, Ferrara, Hartke, Jacobson, Kaul, West, 2012)

Let $\pi_1 = (d_1^{(1)}, \ldots, d_n^{(1)})$ and $\pi_2 = (d_1^{(2)}, \ldots, d_n^{(2)})$ be graphic sequences.

If $\Delta = \max\{d_i^{(1)} + d_i^{(2)}\}$ and $\delta = \min\{d_i^{(1)} + d_i^{(2)}\}$
are such that $\Delta \leq \sqrt{2\delta n} - (\delta - 1)$,

then π_1 and π_2 pack, except that strict inequality is required when $\delta = 1$.

This result is sharp for all δ.

Comparing the Results

Sauer–Spencer : $\Delta_1\Delta_2 < n/2 \Rightarrow G_1$ and G_2 pack.

BFHJKW : (with $\delta = 1$)

$$\max\{d_i^{(1)} + d_i^{(2)}\} < \sqrt{2n} \Rightarrow \pi_1 \text{ and } \pi_2 \text{ pack.}$$

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$$\Delta_1 + \Delta_2 < \sqrt{2n} \Rightarrow \Delta_1 \Delta_2 < n/2$$

A Direct Analogue to Sauer-Spencer

We conjecture the following, which would be a more direct analogue to the Sauer-Spencer Theorem.

Conjecture

Let $\pi_1 = (d_1^{(1)}, \ldots, d_n^{(1)})$ and $\pi_2 = (d_1^{(2)}, \ldots, d_n^{(2)})$ be graphic sequences with δ the least entry in $\pi_1 + \pi_2$.

If $\delta \geq 1$ and $\max\{d_i^{(1)} d_i^{(2)}\} < n/2$, then π_1 and π_2 pack.

Kundu's k-Factor Theorem

When necessary conditions are sufficient for packing:

Kundu, 1973

Let k be a positive integer, and let π_1 and π_2 be graphic sequences such that each term in π_2 is k.

Then π_1 and π_2 pack if and only if $\pi_1 + \pi_2$ is graphic.

Alternatively, if $\pi = (d_1, \ldots, d_n)$ is a graphic sequence such that $\pi - k = (d_1 - k, \ldots, d_n - k)$ is graphic, then there exists a realization G of π that has a k-factor.

Recall, k-factor is a k-regular spanning subgraph.

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Extending Kundu's Theorem

Rao and Rao showed the following while attempting to prove the (then) k-factor conjecture.

A.R. Rao and S.B. Rao, 1972

Let $\pi = (d_1, \ldots, d_n)$ be a graphic sequence such that $\pi - k = (d_1 - k, \ldots, d_n - k)$ is graphic for some $k > 0$.

Then for any nonnegative integer $r \leq k$ such that rn is even, $\pi - r = (d_1 - r, \ldots, d_n - r)$ is also graphic.

Therefore, if some realization of π has a k-factor, then there is also a realization that contains an r-factor for any (feasible) $r < k$.

A Conjecture

We conjecture that Kundu's Theorem can be strengthened in the following manner.

Conjecture

Let $k > 0$ and let $\pi = (d_1, \ldots, d_n)$ be a graphic sequence such that $\pi - k = (d_1 - k, \ldots, d_n - k)$ is graphic.

Then for any $k_1, \ldots, k_t$ such that nk_i is even for all i and

$$k_1 + k_2 + \ldots k_t = k,$$

there is a realization G of π containing edge-disjoint subgraphs $F_1, \ldots, F_t$ such that each F_i is a k_i-factor of G.

In other words, there is a realization G of π containing a k-factor that can be decomposed into k_i-factors.

Odd Order

If n is odd, then each k_i (and hence k) must be even.

Since any $2r$-regular graph has a 2-factorization, the conjecture for n odd follows from Kundu's Theorem.

It would therefore be sufficient to prove the following:

Conjecture

Let n be even and let $\pi = (d_1, \ldots, d_n)$ be a graphic seq such that $\pi - k = (d_1 - k, \ldots, d_n - k)$ is graphic for some $k > 0$.

Then there exists a realization G of π that contains k edge-disjoint 1-factors.

We recently learnt that this was originally conjectured by R. Brualdi in 1976. No progress has been reported so far.

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Bounded Maximum Degree

It is straightforward to verify the conjecture when the largest term in π is bounded.

Theorem (Busch, Ferrara, Hartke, Jacobson, Kaul, West, 2011)

Let n be even and let $\pi = (d_1, \ldots, d_n)$ be a graphic sequence such that $\pi - k = (d_1 - k, \ldots, d_n - k)$ is graphic for some $k > 0$

$$\text{and } \max d_i \leq \frac{n}{2} + 1.$$

Then there exists a realization G of π that contains k edge-disjoint 1-factors.

(k-2)-factor and 1-factors

Theorem (Busch, Ferrara, Hartke, Jacobson, Kaul, West, 2012)

Let n be even and let $\pi = (d_1, \ldots, d_n)$ be a graphic sequence such that $\pi - k = (d_1 - k, \ldots, d_n - k)$ is graphic for some $k > 0$

Then there exists a realization of π containing 1-factors F_1 and F_2, and a $(k-2)$-factor F_{k-2} that are edge-disjoint.

The proof utilizes the Gallai-Edmonds decomposition and edge exchanges (2-switches), plus some new ideas.

Conjecture for $k \leq 3$

As a consequence we get that the conjecture is true for $k \leq 3$

Theorem (Busch, Ferrara, Hartke, Jacobson, Kaul, West, 2012)

Let n be even and let $\pi = (d_1, \ldots, d_n)$ be a graphic sequence such that $\pi - 3 = (d_1 - 3, \ldots, d_n - 3)$ is graphic

Then there exists a realization of π containing three edge-disjoint 1-factors.

The Conjectures

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Let $\pi_1 = (d_1^{(1)}, \ldots, d_n^{(1)})$ and $\pi_2 = (d_1^{(2)}, \ldots, d_n^{(2)})$ be graphic sequences with δ the least entry in $\pi_1 + \pi_2$.

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Then there exists a realization G of π that contains k edge-disjoint 1-factors.

Question: What about near-packings?

Other Topics/ Open Problems

Queue-number of Planar graphs?

Chromatic number of graph with bounded Queue-number?

Number of vertex guards needed for Orthogonal Art Galleries with holes?

(using graph coloring so far).

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