The Gap Between the List-Chromatic and Chromatic Numbers

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Joint work with
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List Coloring

List coloring was introduced independently by Vizing (1976) and Erdős, Rubin, and Taylor (1979), as a generalization of usual graph coloring.

For graph $G$ suppose each $v \in V(G)$ is assigned a list, $L(v)$, of colors. We refer to $L$ as a list assignment. An acceptable L-coloring for $G$ is a proper coloring, $f$, of $G$ such that $f(v) \in L(v)$ for all $v \in V(G)$.

When an acceptable L-coloring for $G$ exists, we say that $G$ is L-colorable or L-choosable.
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List Chromatic Number

- The list chromatic number of a graph $G$, written $\chi_\ell(G)$, is the smallest $k$ such that $G$ is $L$-colorable whenever $|L(v)| \geq k$ for each $v \in V(G)$.

- When $\chi_\ell(G) = k$ we say that $G$ has list chromatic number $k$ or that $G$ is $k$-choosable.

- We immediately have that if $\chi(G)$ is the typical chromatic number of a graph $G$, then
  \[ \chi(G) \leq \chi_\ell(G). \]

- A graph is chromatic choosable if $\chi(G) = \chi_\ell(G)$. But we know the gap between $\chi(G)$ and $\chi_\ell(G)$ can be arbitrarily large.
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A Motivating Result

Theorem (Folklore, 1970s)
\[ \chi_\ell(K_{a,b}) = a + 1 \text{ if and only if } b \geq a^a \]

- When \( b \geq a \), we know \( \chi_\ell(K_{a,b}) \leq \text{Col}(K_{a,b}) = a + 1 \).
- So, for fixed \( a \), this theorem tells us the smallest value of \( b \) such that \( \chi_\ell(K_{a,b}) \) is as large as possible (i.e., far from being chromatic-choosable).

We can construct a sequence of graphs with increasing list chromatic number starting from chromatic number 2: \[ \chi(K_{a,a^a}) = \chi(K_{1,1}) = 2 = \chi_\ell(K_{1,1}) < 3 = \chi_\ell(K_{2,4}) < 4 = \chi_\ell(K_{3,27}) < \ldots < a + 1 = \chi_\ell(K_{a,a^a}) \]

**Question:** Can we construct such a sequence starting from chromatic number \( k > 2 \)?
We will give an answer motivated by the Theorem above.
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**Cartesian Product of Graphs**

- The **Cartesian Product** $G \square H$ of graphs $G$ and $H$ is a graph with vertex set $V(G) \times V(H)$.
  
  Two vertices $(u, v)$ and $(u', v')$ are adjacent in $G \square H$ if either $u = u'$ and $vv' \in E(H)$ or $uu' \in E(G)$ and $v = v'$.

- Here’s $C_5 \square P_3$:

![Graph Diagram]

- Every connected graph has a unique factorization under the Cartesian product (that can be found in linear time and space).

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$\chi(G \Box H) = \max\{\chi(G), \chi(H)\}$. 
Theorem (Borowiecki, Jendrol, Kral, Miskuf (2006))

\[ \chi_\ell(G \Box H) \leq \min\{\chi_\ell(G) + \text{Col}(H), \text{Col}(G) + \chi_\ell(H)\} - 1 \]

An easy inductive argument proves this theorem.

For fixed \( G, a: \)

\[ \chi_\ell(G \Box K_{a,b}) \leq \chi_\ell(G) + \text{Col}(K_{a,b}) - 1 = \chi_\ell(G) + a \]

**Question:** Does there always exist a \( b \) such that this upper bound is attained?
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\[ \chi_\ell(G \square K_{a,b}) = \chi_\ell(G) + a, \text{ whenever } b \geq (\chi_\ell(G) + a - 1)^{a|V(G)|} \]

**Question:** Can we improve the lower bound on \( b \)?

**Question:** For which graphs \( G \), can we give a characterization of such \( b \)?

The folklore theorem from earlier gives the characterization when \( G = K_1 \).

- Our main tools are list color function and strongly chromatic choosable graphs.
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The List Color Function

- For $k \in \mathbb{N}$, let $P(G, k)$ denote the number of proper colorings of $G$ with colors from $\{1, \ldots, k\}$.

- It is known that $P(G, k)$ is a polynomial in $k$ of degree $|V(G)|$. We call $P(G, k)$ the chromatic polynomial of $G$.

- The list color function of $G$, $P^\ell(G, k)$, is the minimum number of $k$-list colorings of $G$ where the minimum is taken over all $k$-list assignments for $G$.

- Recall, $P(K_{2,4}, 2) = 2$, and yet $P^\ell(K_{2,4}, 2) = 0$.
- For every graph $G$ and each $k \in \mathbb{N}$, $P^\ell(G, k) \leq P(G, k)$. 
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Some Results on the List Color Function

Theorem (Kostochka and Sidorenko (1990))

If $G$ is a chordal graph (i.e. all cycles contained in $G$ with 4 or more vertices have a chord), then $P_\ell(G, k) = P(G, k)$ for each $k \in \mathbb{N}$.

$P_\ell(G, k)$ need not be a polynomial.

Theorem (Thomassen (2009))

For any graph $G$, $P_\ell(G, k) = P(G, k)$ provided $k > |V(G)|^{10}$.

Theorem (Wang, Qian, Yan (2017))

For any connected graph $G$ with $m$ edges, $P_\ell(G, k) = P(G, k)$ provided $k > \frac{m-1}{\ln(1+\sqrt{2})} \approx 1.135(m - 1)$. 
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First Result

**Theorem (K. and Mudrock)**

\[ \chi_{\ell}(G \square K_{a,b}) = \chi_{\ell}(G) + a, \text{ whenever } b \geq (P_{\ell}(G, \chi_{\ell}(G) + a - 1))^a \]

- If \( G \) has at least one edge, then
  \[ P_{\ell}(G, \chi_{\ell}(G) + a - 1) < (\chi_{\ell}(G) + a - 1)|V(G)|; \] giving a (significant) improvement over the Borowiecki et al. bound.

- We can in fact prove:

  **Theorem (K. and Mudrock)**
  
  *Suppose \( H \) is a bipartite graph with partite sets \( A \) and \( B \) where \( |A| = a \) and \( |B| = b \). Let \( \delta = \min_{v \in B} d_H(v) \). If \( b \geq (P_{\ell}(G, \chi_{\ell}(G) + \delta - 1))^a \), then \( \chi_{\ell}(G \square H) \geq \chi_{\ell}(G) + \delta \).*
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Beyond First Result

Theorem (K. and Mudrock (2018+))

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**Question:** When is this bound sharp? Can we find graphs \( G \) such this bound characterizes \( \chi_\ell(G \square K_{a,b}) = \chi_\ell(G) + a \)?
Beyond First Result

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**Strong Chromatic Choosability**

- List assignment, \( L \), for \( G \) is a **bad \( k \)-assignment** for \( G \) if \( G \) is not \( L \)-colorable and \( |L(v)| = k \) for each \( v \in V(G) \).

- List assignment, \( L \), is **constant** if \( L(v) \) is the same for each \( v \in V(G) \).

- A constant (and bad) 2-assignment for a \( C_5 \):

  ![C_5 Graph](image)

- A graph \( G \) is said to be **strong \( k \)-chromatic choosable** if \( \chi(G) = k \) and every bad \( (k - 1) \)-assignment for \( G \) is constant.
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![Diagram of a cycle graph $C_5$ with list assignments for each vertex.]

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Proposition (K. and Mudrock, 2018+)
Let $G$ be a strong $k$-chromatic choosable graph. Then
(i) $\chi(G) = k = \chi_\ell(G)$ (i.e. $G$ is chromatic choosable),
(ii) $\chi(G - \{v\}) \leq \chi_\ell(G - \{v\}) < k$ for any $v \in V(G)$,
(iii) $k = 2$ if and only if $G$ is $K_2$,
(iv) $k = 3$ if and only if $G$ is an odd cycle,
(v) $G \lor K_p$ is strong $(k + p)$-chromatic choosable for any $p \in \mathbb{N}$.

We essentially have a notion of vertex-criticality for chromatic-choosability.

There are many infinite families of graphs that satisfy this notion.
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Second Result

Theorem (K. and Mudrock)
\[\chi_\ell(G\Box K_{a,b}) = \chi_\ell(G) + a, \text{ whenever } b \geq (P_\ell(G, \chi_\ell(G) + a - 1))^a\]

Theorem (K. and Mudrock)
If \(G\) is a strong \(k\)-chromatic choosable graph and \(k \geq a + 1\), then \(\chi_\ell(G\Box K_{a,b}) = \chi_\ell(G) + a\) if and only if \(b \geq (P_\ell(G, \chi_\ell(G) + a - 1))^a\).

The proof idea is:
If \(L\) is a \((\chi_\ell(G) + a - 1)\)-assignment for \(G\Box K_{a,b}\), there is at most one proper \(L\)-coloring of the copies of \(G\) corresponding to the partite set of size \(a\) that leads to a bad assignment for a given “bottom” copy of \(G\).
We show if two such colorings existed, we could obtain a proper \(a\)-coloring of \(G\).
A simple counting argument completes the proof that there exists a proper \(L\)-coloring of \(G\Box K_{a,b}\) when \(b < (P_\ell(G, \chi_\ell(G) + a - 1))^a\).
Second Result

Theorem (K. and Mudrock)
\[ \chi_{\ell}(G \Box K_{a,b}) = \chi_{\ell}(G) + a, \text{ whenever } b \geq (P_{\ell}(G, \chi_{\ell}(G) + a - 1))^a \]

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The proof idea is:

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Theorem (K. and Mudrock)
\[ \chi_\ell(G \boxtimes K_{a,b}) = \chi_\ell(G) + a, \text{ whenever } b \geq (P_\ell(G, \chi_\ell(G) + a - 1))^a \]

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Corollaries to Second Result

Theorem (K. and Mudrock)

*If* $G$ *is a strong* $k$-chromatic choosable graph and $k \geq a + 1$, *then* $\chi_\ell(G \Box K_{a,b}) = \chi_\ell(G) + a$ *if and only if* 
$$b \geq (P_\ell(G, \chi_\ell(G) + a - 1))^a.$$ 

Corollary (K. and Mudrock)

$\chi_\ell(C_{2t+1} \Box K_2, b) = 5$ *if and only if* 
$$b \geq (P_\ell(C_{2t+1}, 4))^2 = (3^{2t+1} - 3)^2 = 9(9^t - 1)^2.$$ 

Corollary (K. and Mudrock)

*For* $n \geq a + 1$, $\chi_\ell(K_n \Box K_{a,b}) = n + a$ *if and only if* 
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If $G$ is a strong $k$-chromatic choosable graph and $k \geq a + 1$, then $\chi_\ell(G \Box K_{a,b}) = \chi_\ell(G) + a$ if and only if $b \geq (P_\ell(G, \chi_\ell(G) + a - 1))^a$.

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If $G$ is a strong $k$-chromatic choosable graph and $k \geq a + 1$, then $\chi^\ell(G \Box K_{a,b}) = \chi^\ell(G) + a$ if and only if $b \geq (P^\ell(G, \chi^\ell(G) + a - 1))^a$.

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This corollary shows the bound in the Theorem is sharp for all $a$.

We can construct an arbitrarily long sequence of graphs with increasing list chromatic number starting from chromatic number $n$:

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\chi(K_n \Box K_{a,b}) = \chi(K_n) = n = \chi_\ell(K_n \Box K_{0,1}) < n + 1 = \chi_\ell(K_n \Box K_{1,n!}) < n + 2 = \chi_\ell(K_n \Box K_{2,((n+1)!)^2}) < \ldots
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Corollary (K. and Mudrock, 2018+)

Let $G$ be a strong $k$-chromatic choosable graph. Then,

$$\chi_\ell(G \Box K_{1,s}) = \begin{cases} k & \text{if } s < P_\ell(G, k) \\ k + 1 & \text{if } s \geq P_\ell(G, k). \end{cases}$$
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Corollary (K. and Mudrock, 2018+)

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\chi_\ell(C_{2t+1} \square K_1,s) = \begin{cases} 
  3 & \text{if } s < 2^{2t+1} - 2 \\
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Corollary (K. and Mudrock, 2018+)

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\chi_\ell(K_n \square K_1,s) = \begin{cases} 
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$$

Corollary (K. and Mudrock, 2018+)

$$
\chi_\ell((K_n \lor C_{2t+1}) \square K_1,s) = \begin{cases} 
  n + 3 & \text{if } s < \frac{1}{3}(n + 3)!(4^t - 1) \\
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Extending the Second Result

Theorem (K. and Mudrock)

If $G$ is a strong $k$-chromatic choosable graph and $k \geq a + 1$, then
\[ \chi_\ell(G \Box K_{a,b}) = \chi_\ell(G) + a \text{ if and only if } b \geq (P_\ell(G, \chi_\ell(G) + a - 1))^a. \]

Open Question: Can we remove the $k \geq a + 1$ in the above theorem?

Theorem (K. and Mudrock)

If $G$ is a strong $k$-chromatic choosable graph, then
\[ \chi_\ell(G \Box K_{a,b}) < \chi_\ell(G) + a \text{ whenever } b < (P_\ell(G, \chi_\ell(G) + a - 1))^a/2^{k-1}. \]
Extending the Second Result

Theorem (K. and Mudrock)

If $G$ is a strong $k$-chromatic choosable graph and $k \geq a + 1$, then $\chi_\ell(G \square K_{a,b}) = \chi_\ell(G) + a$ if and only if $b \geq (P_\ell(G, \chi_\ell(G) + a - 1))^a$.

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Questions?

- Define $f_a(G)$ as the smallest $b$ s.t. $\chi_\ell(G \square K_{a,b}) = \chi_\ell(G) + a$.
- For what graphs does $f_a(G) = (P_\ell(G, \chi_\ell(G) + a - 1))^a$?
- Does there exist a strongly chromatic-choosable graph $M$ such that $f_a(M) < (P_\ell(M, \chi_\ell(M) + a - 1))^a$? Or, can we remove the condition $k \geq a + 1$ in the second theorem?
- Is it the case that $f_a(K_n) = \left(\frac{(n+a-1)!}{(a-1)!}\right)^a$ for each $n, a$?
- We can ask the above question for any family of strongly chromatic-choosable graphs.

- Is it always the case that $P_\ell(G, k) = P(G, k)$ when $G$ is strong chromatic choosable?
- (Thomassen 2009) Does there exist a graph $G$ and a natural number $k > 2$ such that $P_\ell(G, k) = 1$?
- (Mohar 2001) Let $G$ be a $(\Delta(G) + 1)$-edge-critical graph. Then prove that $L(G)$ is strong $(\Delta(G) + 1)$-chromatic choosable.
Define $f_a(G)$ as the smallest $b$ s.t. $\chi_\ell(G \Box K_{a,b}) = \chi_\ell(G) + a$.

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