Reliable Error Estimation for Quasi-Monte Carlo Methods

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Outline

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Multidimensional numerical integration?

- Applications: option pricing, ion transport models, statistical physics...

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Some common techniques:

Method	Convergence
Trapezoidal rule:	$\mathcal{O}(n^{-2/d})$
Simpson's rule:	$\mathcal{O}(n^{-4/d})$
Monte Carlo:	$\mathcal{O}(n^{-1/2})$
Quasi-Monte Carlo:	$\mathcal{O}(n^{-1+\varepsilon})$

(n: number of data points)

Computational problem

Quasi-Monte Carlo methods:

$$\int_{[0,1)^d} f(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x} \approx \frac{1}{n}\sum_{i=0}^{n-1} f(\boldsymbol{z}_i)\,, \qquad \text{for fixed} \quad \boldsymbol{z}_0, \boldsymbol{z}_1, \dots \in [0,1)^d$$

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<u>KOKSMA-HLAWKA INEQUALITY</u>

$$\operatorname{err}(f, b^m, \{\boldsymbol{z}_i\}) := \left| \int_{[0,1)^d} f(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x} - \frac{1}{n}\sum_{i=0}^{n-1} f(\boldsymbol{z}_i) \right| \leqslant \overbrace{\underbrace{D^*(b^m, \{\boldsymbol{z}_i\})}_{\mathcal{O}(n^{-1+\varepsilon})}}^{\text{Quality of } \boldsymbol{z}_i} \quad \overbrace{V(f)}^{\text{Roughness of } f}$$

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Rank-1 Lattice Points

Rank-1 Lattice Points are of the form:

$$\mathcal{P}_m := \left\{ \boldsymbol{h} \frac{j}{b^m} \pmod 1 \, ; \, j = 0, \dots, b^m - 1 \right\}$$

We provide the following structure:

$$\mathcal{P}_m := \{\boldsymbol{z}_i\}_{i=0}^{b^m-1},$$
$$\{0\} = \mathcal{P}_0 \subseteq \cdots \subseteq \mathcal{P}_m \subseteq \cdots \subseteq \mathcal{P}_\infty := \{\boldsymbol{z}_i\}_{i=0}^{\infty}$$
$$\mathcal{P}_m^\perp := \{\boldsymbol{k} \in \mathbb{Z}^d : \langle \boldsymbol{k}, \boldsymbol{z}_i \rangle = 0, \;\; i = 0, \dots b^m - 1\},$$
$$\mathbb{Z}^d = \mathcal{P}_0^\perp \supseteq \cdots \supseteq \mathcal{P}_m^\perp \supseteq \cdots \supseteq \mathcal{P}_\infty^\perp = \{0\}$$

Example of Rank-1 Lattice

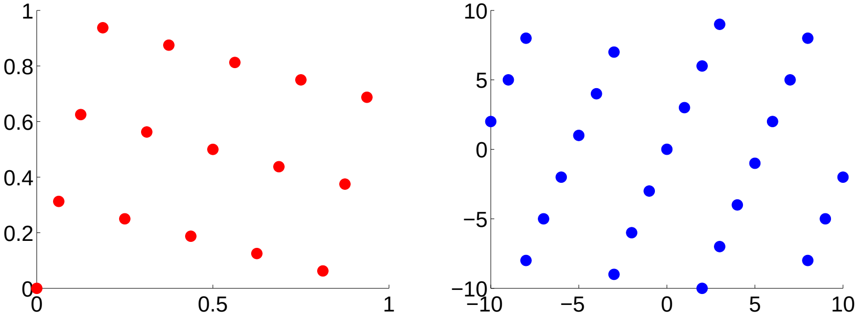
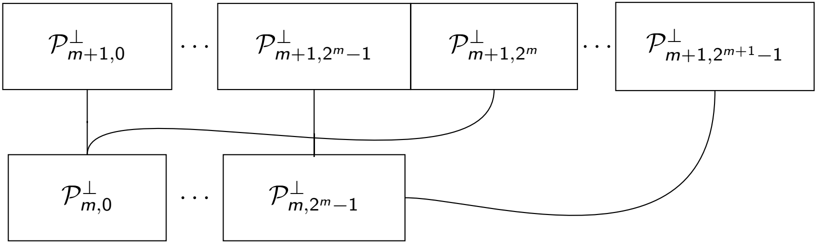


Figure: Lattice example for $\boldsymbol{h} = (1,5)^T$ and $n = 16$ in base 2. On the left, $\mathcal{P}_4$; on the right, some points of $\mathcal{P}_4^{\perp}$.

Ordering wavenumbers

We define a bijective mapping $\tilde{\boldsymbol{k}} : \mathbb{N}_0 \to \mathbb{Z}^d$ such that for all $m, \lambda \in \mathbb{N}_0$ and $\kappa = 0, \dots, b^m - 1$,

$$\tilde{\boldsymbol{k}}(0) = \boldsymbol{0}, \qquad \tilde{\boldsymbol{k}}(\kappa + \lambda b^m) = \tilde{\boldsymbol{k}}(\kappa) + \boldsymbol{l} \text{ for some } \boldsymbol{l} \in \mathcal{P}_m^\perp.$$



With this mapping, the Fourier coefficients are ordered in $\mathbb{N}_0$ and one can rewrite $\hat{f}_\kappa := \hat{f}(\tilde{\boldsymbol{k}}(\kappa))$.

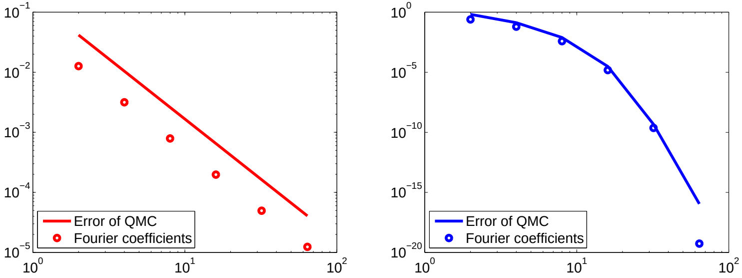
Error in terms of Fourier coefficients

The error is then:

$$\begin{aligned}\operatorname{err}(f, b^m, \{\boldsymbol{z}_i\}) := & \left| \int_{[0,1)^d} f(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x} - \frac{1}{b^m} \sum_{i=0}^{b^m-1} f(\boldsymbol{z}_i) \right| \\ = & \left| \hat{f}(\boldsymbol{0}) - \tilde{f}_m(\boldsymbol{0}) \right| \\ = & \left| \sum_{\boldsymbol{l} \in \mathcal{P}_m^\perp \setminus \{\boldsymbol{0}\}} \hat{f}(\boldsymbol{l}) \right| \\ = & \left| \sum_{\lambda=1}^{\infty} \hat{f}_{\lambda b^m} \right|\end{aligned}$$

Intuition

$$f(x) = x^2 - x + 1/4 \qquad g(x) = \frac{3}{5 - 4\cos(2\pi x)}$$
$$\hat{f}(k) = \frac{1}{2\pi^2 k^2},\ k \in \mathbb{Z}\backslash\{0\} \qquad \hat{g}(k) = \frac{1}{2^{|k|}},\ k \in \mathbb{Z}\backslash\{0\}$$



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Error bound in terms of the Fourier Coefficients

Cone assumptions ($f \in \mathcal{C} \Leftrightarrow af \in \mathcal{C}$)

There exist non-increasing $\widehat{\omega}$ and $\mathring{\omega}$ such that for all $0 \leqslant \ell \leqslant m$

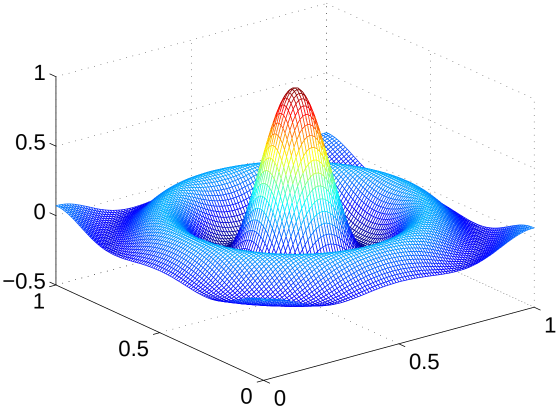
$$\widehat{S}(\ell, m) := \sum_{\kappa=\lfloor b^{\ell-1} \rfloor}^{b^\ell-1} \sum_{\lambda=1}^{\infty} \left|\hat{f}_{\kappa+\lambda b^m}\right|, \quad \check{S}(m) := \sum_{\kappa=b^m}^{\infty} \left|\hat{f}_\kappa\right| \quad S(\ell) := \sum_{\kappa=\lfloor b^{\ell-1} \rfloor}^{b^\ell-1} \left|\hat{f}_\kappa\right|$$

$$\widehat{S}(\ell, m) \leqslant \widehat{\omega}(m-\ell)\check{S}(m) \quad \forall \ell, \qquad \check{S}(m) \leqslant \mathring{\omega}(m-\ell)S(\ell) \quad \forall \ell_* \leqslant \ell.$$

Then we can bound the error as follows:

$$\mathrm{err}(f, b^m, \{\boldsymbol{z}_i\}) \leqslant \sum_{\lambda=1}^{\infty} \left|\hat{f}_{\lambda b^m}\right| = \widehat{S}(0, m) \leqslant \widehat{\omega}(m)\mathring{\omega}(m-\ell)S(\ell).$$

Example of cone assumption



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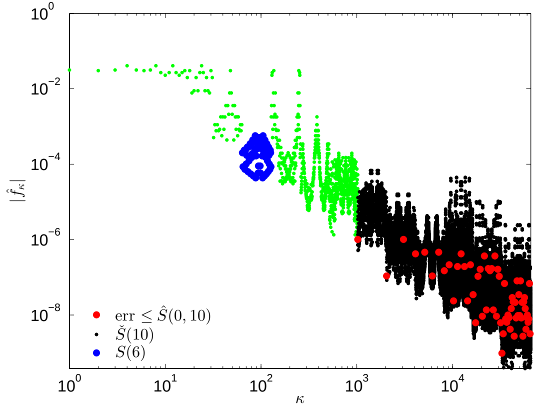

Error bound in terms of the Approximated Fourier Coefficients

Since $S(\ell)$ is not an input for the error bound, we need to bound it in terms of the approximated Fourier coefficients:

$$\begin{aligned}
S(\ell) &= \sum_{\kappa=\lfloor b^{\ell-1}\rfloor}^{b^\ell-1} \left|\hat{f}_\kappa\right| = \sum_{\kappa=\lfloor b^{\ell-1}\rfloor}^{b^\ell-1} \left|\tilde{f}_{m,\kappa} - \sum_{\lambda=1}^{\infty} \hat{f}_{\kappa+\lambda b^m}\right| \\
&\leqslant \underbrace{\sum_{\kappa=\lfloor b^{\ell-1}\rfloor}^{b^\ell-1} \left|\tilde{f}_{m,\kappa}\right|}_{\widetilde{S}(\ell,m)} + \underbrace{\sum_{\kappa=\lfloor b^{\ell-1}\rfloor}^{b^\ell-1} \sum_{\lambda=1}^{\infty} \left|\hat{f}_{\kappa+\lambda b^m}\right|}_{\widehat{S}(\ell,m)} \\
&\leqslant \widetilde{S}(\ell,m) + \widehat{\omega}(m-\ell)\breve{\omega}(m-\ell)S(\ell)
\end{aligned}$$

$$S(\ell) \leqslant \frac{\widetilde{S}(\ell,m)}{1-\widehat{\omega}(m-\ell)\breve{\omega}(m-\ell)} \qquad \text{whenever } \widehat{\omega}(m-\ell)\breve{\omega}(m-\ell) < 1.$$

Guaranteed, Adaptive and Automatic algorithm

Given an error tolerance $\varepsilon > 0$ and an integrand f, fix $r \in \mathbb{N}$ and let

$$\mathfrak{C} = \frac{\breve{\omega}(r)}{1 - \widehat{\omega}(r)\breve{\omega}(r)}.$$

Initialize $m = r + \ell^* \in \mathbb{N}$.

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Step 2. If the error tolerance is satisfied, i.e.

$$\mathfrak{C}\widehat{\omega}(m)\widetilde{S}(m-r, m) \leqslant \varepsilon,$$

then return the Rank-1 Lattice answer.

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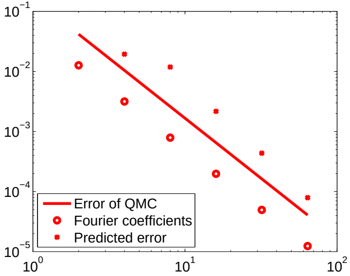
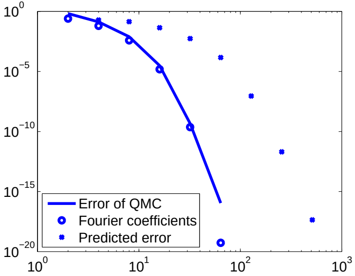
<u>Theorem:</u> $\left| \int_{[0,1)^d} f(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x} - \frac{1}{b^m}\sum_{i=1}^{m} f(\boldsymbol{z}_i) \right| \leqslant \varepsilon$ if f is in the cone.

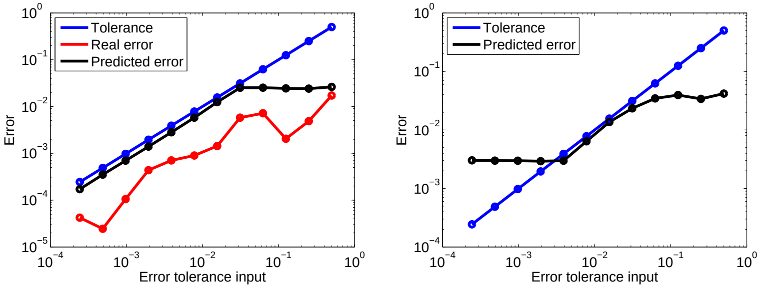
Outline

Going back to the beginning...





Basket option: European Call in 1-D and 10-D



1-D	10-D
$S_0 = 100,\ K = 120,$ $\sigma = 0.2,\ r = 0.05,\ T = 1$	$S_0^{(i)} = 100,\ K = 1200,$ σ random, $r = 0.05,\ T = 1$

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Future work

- Defining the relative error bound.
- Fitting the cone of functions into a more familiar space (Korobov).
- Lower and upper bounds on computational complexity.
- Implementing and adding this code to GAIL (`http://code.google.com/p/gail/`).

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