Adaptive Quasi-Monte Carlo Where Each Integrand Value Depends on All Data Sites: the American Option

Lluís Antoni Jiménez Rugama
Joint work with Fred J. Hickernell (IIT) and Da Li (IIT)

Room 120, Bldg E1, Department of Applied Mathematics
Illinois Institute of Technology, Chicago, 60616 IL
Email: ljimene1@hawk.iit.edu

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Outline

- American Options
- Quasi-Monte Carlo Cubatures
- The Challenge
- Improving Cubature Efficiency
- Conclusions and Future Work

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- American Options—The American option and the Longstaff and Schwartz (2001) pricing method.
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The American Put Option

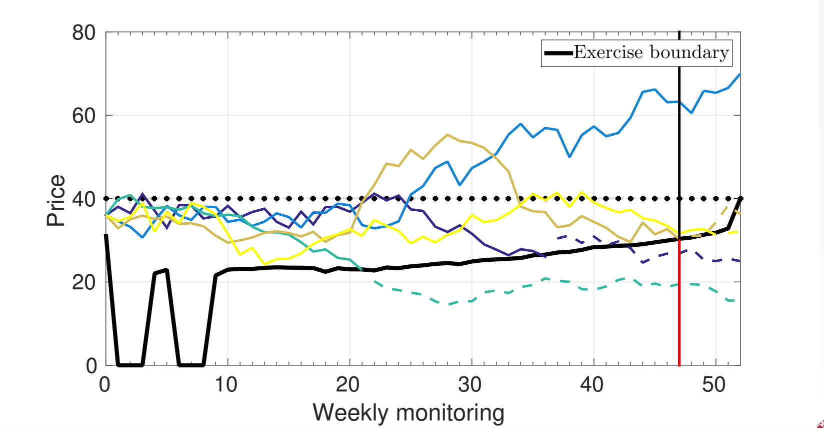
Important parameters:

- Strike price K
- Maturity T
- Exercise time $t \in [0, T]$
- Payoff $\max(K - S(t), 0)$

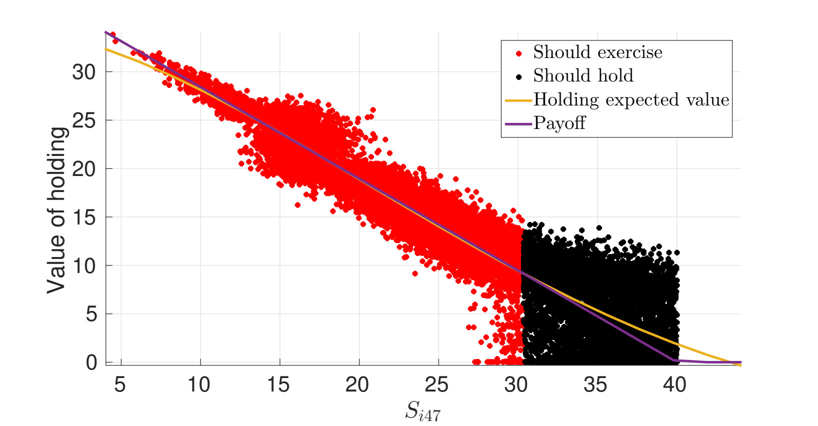
For our examples we will use:

- $K = 40$
- $T = 1$ year
- Weekly monitoring, $[1/52, 2/52, \ldots, 1]$
- Geo. Brownian motion with $S_0 = 36$, $r = 6\%$, and $\sigma = 50\%$

The Longstaff and Schwartz (2001) Method



Regression at Week $j = 47$



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Low Discrepancy Sequences

Consider an embedded sequence in base b,

$$\mathcal{P}_0 = \{\mathbf{0}\} \subset \cdots \subset \mathcal{P}_m = \{\boldsymbol{x}_i\}_{i=0}^{b^m} \subset \cdots \subset \mathcal{P}_\infty = \{\boldsymbol{x}_i\}_{i=0}^{\infty}.$$

Each $\mathcal{P}_m$ forms a group:

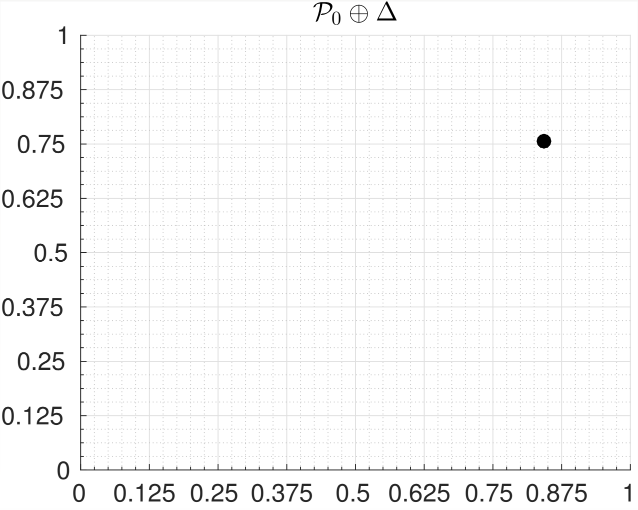
- Digital nets

$$\boldsymbol{x} \oplus \boldsymbol{t} = \left(\sum_{\ell=1}^{\infty} [(x_{j\ell} + t_{j\ell}) \bmod b] b^{-\ell} \pmod 1 \right)_{j=1}^{d}.$$

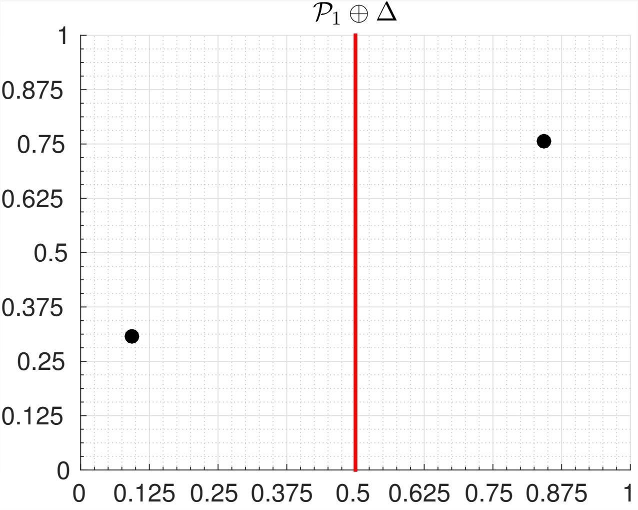
- Rank-1 lattices

$$\boldsymbol{x} \oplus \boldsymbol{t} = (\boldsymbol{x} + \boldsymbol{t}) \bmod 1.$$

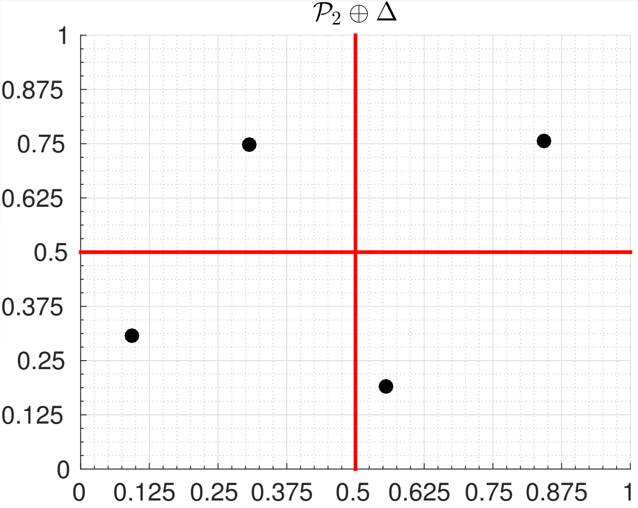
Scrambled and Digitally Shifted Sobol' Sequence



Scrambled and Digitally Shifted Sobol' Sequence



Scrambled and Digitally Shifted Sobol' Sequence



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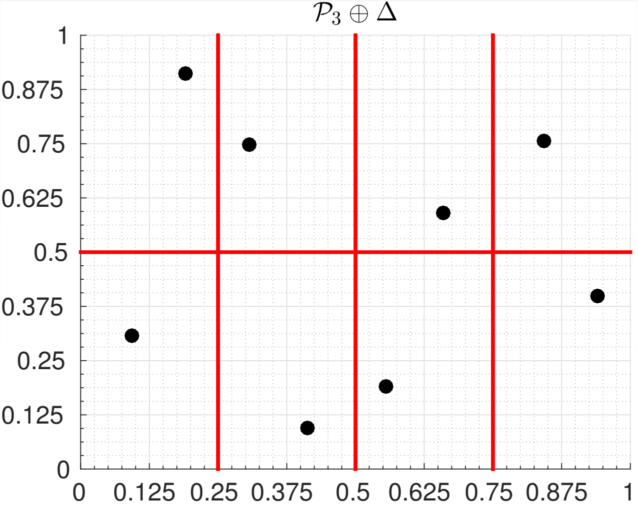
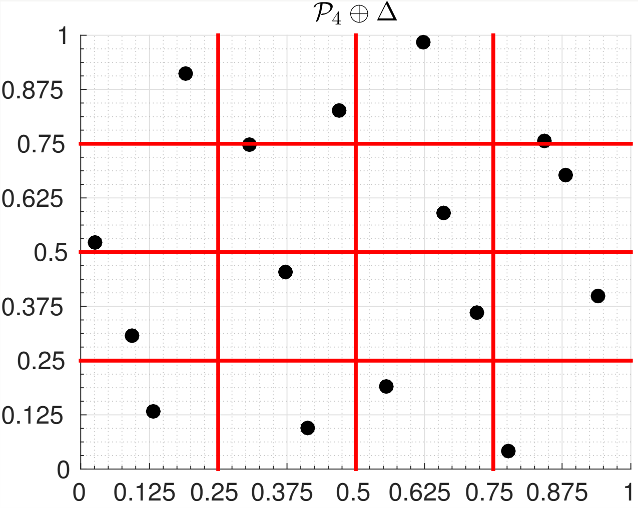
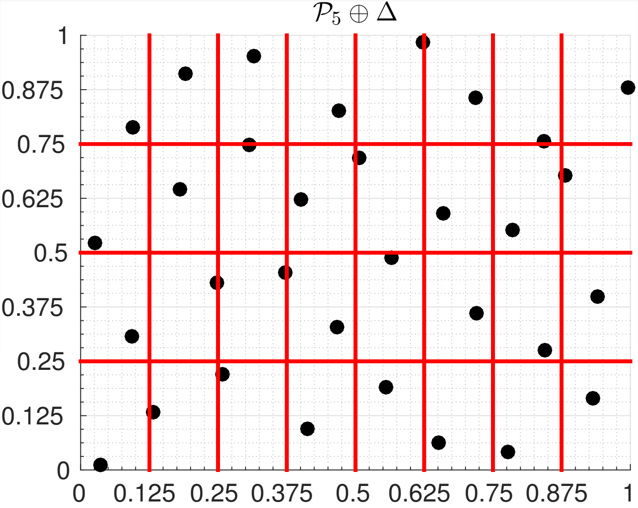


Scrambled and Digitally Shifted Sobol' Sequence

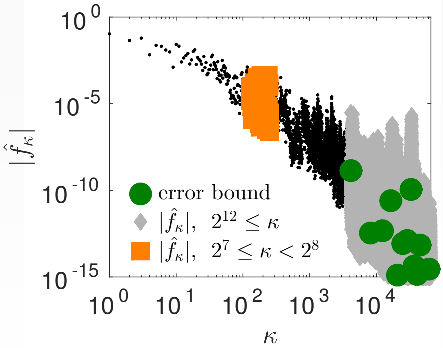


Scrambled and Digitally Shifted Sobol' Sequence



Adaptive Algorithm for $f \in \mathcal{C}$

$$\left| \int_{[0,1)^d} f(\boldsymbol{x})\, d\boldsymbol{x} - \frac{1}{|\mathcal{P}_m|} \sum_{\boldsymbol{x} \in \mathcal{P}_m} f(\boldsymbol{x}) \right| \leqslant \overbrace{\sum \bullet}^{\text{Dual net/lat Fourier coef}} \leqslant \mathfrak{C}(r,m) \sum_{\kappa=\lfloor b^{m-r-1} \rfloor}^{b^{m-r}-1} \left| \tilde{f}_{m,\kappa} \right| \overset{\text{Want}}{\leqslant} \varepsilon$$

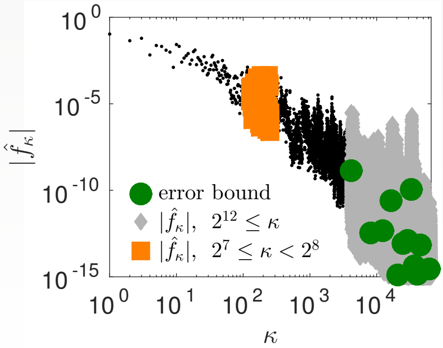


But we

- Do not want to assume that ● decay at a given rate.
- We cannot get ● from data.

Adaptive Algorithm for $f \in \mathcal{C}$

$$\left| \int_{[0,1)^d} f(\boldsymbol{x})\, d\boldsymbol{x} - \frac{1}{|\mathcal{P}_m|} \sum_{\boldsymbol{x} \in \mathcal{P}_m} f(\boldsymbol{x}) \right| \leqslant \overbrace{\sum \bullet}^{\text{Dual net/lat Fourier coef}} \leqslant \mathfrak{C}(r,m) \sum_{\kappa=\lfloor b^{m-r-1} \rfloor}^{b^{m-r}-1} \left| \tilde{f}_{m,\kappa} \right| \overset{\text{Want}}{\leqslant} \varepsilon$$



$$\mathcal{C} = \left\{ \sum \blacksquare \text{ bounds } \sum \blacklozenge \atop \sum \blacklozenge \text{ bounds } \sum \bullet \right\}$$

Automatic Algorithm

We choose an initial number of points b^{m_0}. Then,

Step 1 Generate $\{y_i\}_{i=0}^{b^{m_0}-1} = \{f(\boldsymbol{x}_i)\}_{i=0}^{b^{m_0}-1}$ and compute its discrete Fourier/Walsh coefficients.

Step 2 Estimate the error bound. If it is smaller than ε, STOP.

Step 3 Otherwise, set $m = m_0 + 1$.

- Step 3.1 Generate $\{y_i\}_{i=b^{m-1}}^{b^m-1}$, and update the discrete Fourier/Walsh transform.
- Step 3.2 Update the error bound. If it is smaller than ε, STOP.
- Step 3.3 Increment m by one, and go to Step 3.1.

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Updating the Function Values

The American option payoff computed according to the Longstaff and Schwartz (2001) method is not a regular function over $[0,1]^d$:

$$\mathrm{payoff_i} = f(\boldsymbol{x}_i) \quad \text{becomes} \quad \{\mathrm{payoff_i}\}_{i=0}^{b^m-1} = f\left(\{\boldsymbol{x}_i\}_{i=0}^{b^m-1}\right).$$

Former Automatic Algorithm

We choose an initial number of points b^{m_0}. Then,

Step 1 Generate $\{\text{payoff}_\text{i}\}_{i=0}^{b^{m_0}-1} = \{f(\boldsymbol{x}_i)\}_{i=0}^{b^{m_0}-1}$ and compute its discrete Fourier/Walsh coefficients.

Step 2 Estimate the error bound. If it is smaller than ε, STOP.

Step 3 Otherwise, set $m = m_0 + 1$.

- Step 3.1 Generate $\{\text{payoff}_\text{i}\}_{i=b^{m-1}}^{b^m-1}$, and update the discrete Fourier/Walsh transform.
- Step 3.2 Update the error bound. If it is smaller than ε, STOP.
- Step 3.3 Increment m by one, and go to Step 3.1.

New Automatic Algorithm

We choose an initial number of points b^{m_0}. Then,

Step 1 Generate $\{\mathrm{payoff_i}\}_{i=0}^{b^{m_0}-1} = f(\{\boldsymbol{x}_i\}_{i=0}^{b^{m_0}-1})$ and compute its discrete Fourier/Walsh coefficients.

Step 2 Estimate the error bound. If it is smaller than ε, STOP.

Step 3 Otherwise, set $m = m_0 + 1$.

- Step 3.1 Generate $\{\mathrm{payoff_i}\}_{i=0}^{b^{m}-1} = f(\{\boldsymbol{x}_i\}_{i=0}^{b^{m}-1})$, and recompute the discrete Fourier/Walsh coeff.
- Step 3.2 Update the error bound. If it is smaller than ε, STOP.
- Step 3.3 Increment m by one, and go to Step 3.1.

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Control Variates Introduction

If $I(g) = \int_{[0,1)^d} g(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x}$ is known, then

$$\int_{[0,1)^d} f(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x} = \int_{[0,1)^d} f(\boldsymbol{x}) + \beta(I(g) - g(\boldsymbol{x}))\,\mathrm{d}\boldsymbol{x},$$

and, based on our cubatures, we choose β such that

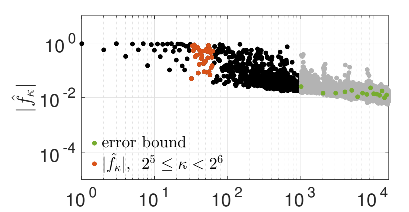
$$\sum_{\kappa=\lfloor b^{m-1}\rfloor}^{b^m-1} \left|\hat{f}_\kappa - \beta\hat{g}_\kappa\right| \overset{m\to\infty}{\Longrightarrow} 0 \quad \text{faster than} \quad \sum_{\kappa=\lfloor b^{m-1}\rfloor}^{b^m-1} \left|\hat{f}_\kappa\right| \overset{m\to\infty}{\Longrightarrow} 0.$$

This choice is not necessarily

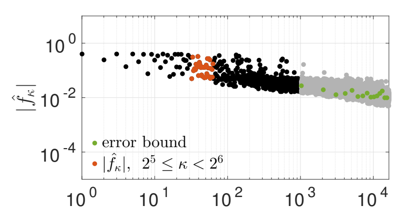
$$\beta = \frac{\operatorname{cov}(f(\boldsymbol{X}), g(\boldsymbol{X}))}{\operatorname{Var}(g(\boldsymbol{X}))} = \operatorname*{argmin}_b \sum_{\kappa=0}^{\infty} \left|\hat{f}_\kappa - b\hat{g}_\kappa\right|^2,$$

as noted by Hickernell et al. (2005).

American Option Walsh Coefficients: Control Variates



Cholesky (time differencing)

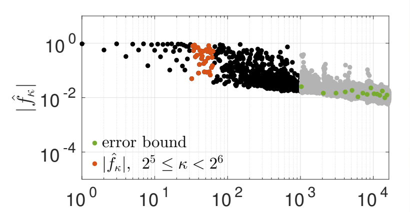


European put control variate

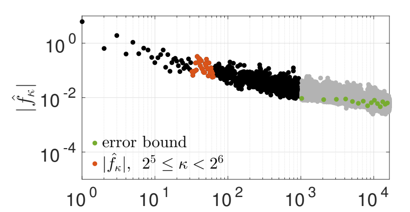
For $\varepsilon = 0.05$,

- $131,072$ points and 3.12 seconds.
- $32,768$ points and 0.75 seconds.

American Option Walsh Coefficients: BM Construction



Cholesky (time differencing)

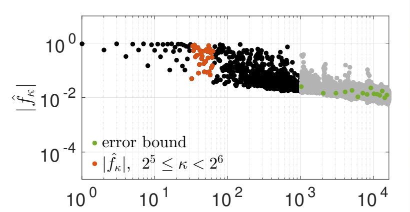


PCA

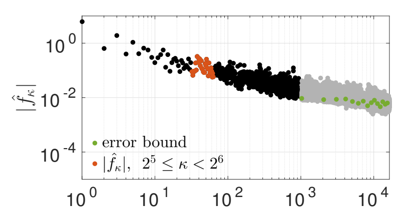
For $\varepsilon = 0.05$,

- $131,072$ points and 3.12 seconds.
- $16,384$ points and 0.44 seconds.

American Option Walsh Coefficients: BM Construction



Cholesky (time differencing)



PCA

For $\varepsilon = 0.05$,

- $131,072$ points and 3.12 seconds.
- $16,384$ points and 0.44 seconds.
- $8,192$ points and 0.28 seconds.

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Future Work

Additional capabilities,

- Cost from $\mathcal{O}(n\log(n))$ to $\mathcal{O}(n\log(n)^2)$, $n=b^m$.
- Allows for shifts and relative error tolerances.

Future work,

- Test the algorithm with rank-1 lattices.
- Apply multilevel quasi-Monte Carlo.
- Design adaptive cone conditions.
- Study other asset price models such as the Heston model.

References I

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Regression Basis

The basis proposed in Longstaff and Schwartz (2001) and used for our examples in our slides is:

- $e^{-\frac{x}{2S_0}}$.
- $e^{-\frac{x}{2S_0}}(1 - \frac{x}{S_0})$.
- $e^{-\frac{x}{2S_0}}(1 - 2\frac{x}{S_0} + \frac{x^2}{2S_0^2})$.

Inside and Outside $\mathcal{C}$