Generalizing the Tolerance Function for Guaranteed Algorithms

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- tol $\in \mathbb{R}^+$, the maximum tolerance we are willing to accept. In other words, we want

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Thus, we increase n until $\widehat{\varepsilon}_n(f) \leqslant tol$.

Example

`cubLattice_g` and `cubSobol_g` algorithms in GAIL (free Matlab library that you can find on `https://code.google.com/p/gail/`)

- $I(f) := \int_{[0,1)^d} f(\boldsymbol{x})\mathrm{d}\boldsymbol{x}$.
- $\widehat{I}_n(f) := \frac{1}{n}\sum_{i=0}^{n-1} f(\boldsymbol{x}_i)$, where $\{\boldsymbol{x}_i\}$ rank-1 lattices or Sobol' points.
- $\widehat{\varepsilon}_n(f)$ defined according to (Jiménez Rugama and Hickernell, 2014) and (Hickernell and Jiménez Rugama, 2014).

Problem to Solve

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For example: $\left|I - \widehat{I}_n(f)\right| \leqslant \varepsilon_r \left|I\right|$.

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Generalization

Consider tol$(a, b) \in \mathbb{R}$ to have the following properties:

- Lipschitz-1 in b.
- Non-decreasing in both arguments, i.e. tol$(a, b) \leqslant$ tol(a', b') for $a \leqslant a'$ and $b \leqslant b'$.

Generalization

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Examples for $a = \varepsilon_a$ and $b = \varepsilon_r |I|$:

- tol$(a, b) = \max(a, b)$
- tol$(a, b) = \theta a + (1 - \theta) b$ with $\theta \in [0, 1]$.

New Algorithm $\widetilde{I}_n$

Define,

$$\widetilde{I}_n := \widehat{I}_n + \Delta_{n,-}$$

with

$$\Delta_{n,\pm} := \frac{1}{2}\left[\mathrm{tol}\left(\varepsilon_a, \varepsilon_r \left|\widehat{I}_n - \widehat{\varepsilon}_n\right|\right) \pm \mathrm{tol}\left(\varepsilon_a, \varepsilon_r \left|\widehat{I}_n + \widehat{\varepsilon}_n\right|\right)\right]$$

Main lemma

Lemma 1

If $\widehat{\varepsilon}_n \leqslant \Delta_{n,+}$ *then*

$$\left| I - \widetilde{I}_n \right| \leqslant \mathit{tol}(\varepsilon_a, \varepsilon_r \left| I \right|).$$

Main lemma

Lemma 1

If $\widehat{\varepsilon}_n \leqslant \Delta_{n,+}$ *then*

$$\left|I - \widetilde{I}_n\right| \leqslant \mathit{tol}(\varepsilon_a, \varepsilon_r \left|I\right|).$$

Note:

- Before, we increased n until $\widehat{\varepsilon}_n \leqslant \text{tol}$ to have,

$$\left|I - \widehat{I}_n\right| \leqslant \text{tol}$$

- Now, we increase n until $\widehat{\varepsilon}_n \leqslant \Delta_{n,+}$ to have,

$$\left|I - \widetilde{I}_n\right| \leqslant \text{tol}(\varepsilon_a, \varepsilon_r \left|I\right|)$$

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Lemma for Complexity

Lemma 2

If

$$\widehat{\varepsilon}_n \leqslant \frac{\mathit{tol}(\varepsilon_a, \varepsilon_r \left|I\right|)}{1 + \varepsilon_r}$$

then,

$$\widehat{\varepsilon}_n \leqslant \Delta_{n,+}$$

Note: Thus, using lemma (1), $\left|I - \widetilde{I}_n\right| \leqslant \mathrm{tol}(\varepsilon_a, \varepsilon_r \left|I\right|)$.

Upper Bound

Define $M(\varepsilon, \mathcal{C}_p)$ such that,

$$n \geqslant M(\varepsilon, \mathcal{C}_p) \Longrightarrow \widehat{\varepsilon}_n \leqslant \varepsilon$$

where $\mathcal{C}_p$ is the set of parameters specifying the functions under which the guarantees hold.

If $M^* = M\left(\frac{\mathrm{tol}(\varepsilon_a, \varepsilon_r |I|)}{1+\varepsilon_r}, \mathcal{C}_p\right)$,

$$n \geqslant M^* \overset{Mdef}{\Longrightarrow} \widehat{\varepsilon}_n \leqslant \frac{\mathrm{tol}\left(\varepsilon_a, \varepsilon_r \,|I|\right)}{1+\varepsilon_r}$$

$$\overset{lem(2)}{\Longrightarrow} \widehat{\varepsilon}_n \leqslant \Delta_{n,+}$$

$$\overset{lem(1)}{\Longrightarrow} \left|I - \widetilde{I}_n\right| \leqslant \mathrm{tol}(\varepsilon_a, \varepsilon_r \,|I|)$$

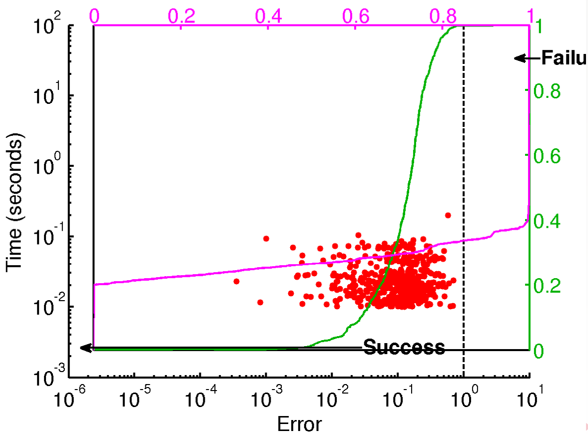
The Quasi-Monte Carlo Example

For the previous example, where we evaluate the function at some points and compute the Fast Fourier/Walsh Transform,

$$\mathrm{cost}\left(\widetilde{I}_n, \mathcal{C}\right) \leqslant cM^* \log(M^*) + \$(f)(M^*)$$

Pricing the Geometric Asian Call

Figure : 500 replications pricing the Geometric Asian Call option for dimensions $1, 2, 4, 8, 16, 32$ and 64 chosen randomly. The tolerance function was $\max\left(10^{-2}, 10^{-2}\,|I|\right)$.



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Future work

- Lower bound on the computational complexity for $\widetilde{I}_n$.
- Fitting the cone of functions into a more familiar space (Korobov).
- Working on Multi-Level quasi Monte Carlo guaranteed hybrid error for infinite dimensional integration. For instance: Asian option pricing in continuous time.
- Implementing and adding this code to GAIL (`http://code.google.com/p/gail/`).

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