Solutions to Assignment #11, version 2. We assume that graphs are simple.

5.1.29 It's easy to find a proper 4-coloring of G, so $\chi(G) \leq 4$. Let u, v be any pair of antipodal vertices, and let $\{x, y\} = V(G) - N(u) - \{u, v\}$; then let $H = G - u - xy$. Copies of $K_4 - e$ in H would force a proper 3-coloring to color v with the same color as x and y; thus H is not 3-colorable. Since $\chi(G) \geq \chi(H) \geq 4$, we have $\chi(G) = \chi(H) = 4$. H must have a 4-critical subgraph H'. Any proper subgraph of H has a vertex of degree less than 3, which cannot be 4-critical; therefore $H' = H$.

5.1.31 (I use "$\times$" for graph product.) Suppose that f is a proper m-coloring of G. Let $S = \{(u, f(u)) : u \in V(G)\}$; then $|S| = n(G)$, and with $V(K_m) = [m]$ we get $S \subseteq V(G \times K_m)$. If $(u, f(u))$ and $(v, f(v))$ are adjacent then since $u \neq v$, we have $uv \in E(G)$ and $f(u) = f(v)$, contradicting that f is a proper coloring of G. Therefore S is an independent set, so $\alpha(G \times K_m) \geq |S| = n(G)$.

Now suppose that $\alpha(G \times K_m) \geq n(G)$. Then there is an independent set $S \subseteq V(G \times K_m)$ of size at least $n(G)$. Any two vertices of $G \times K_m$ with the same first coordinate are adjacent, so for each $v \in V(G)$, S has at most one vertex of the form (v, u). Since $|S| \geq |V(G)|$, there is exactly one such vertex u (for each v), let $f(v) = u$. Observe that f is a proper coloring of G.

Alternatively: Suppose that G is m-colorable. Then $\chi(G) \leq m$, so by Proposition 5.1.11, $\chi(G \times K_m) = \chi(G)$. By Proposition 5.1.7, $\chi(G \times K_m) \geq n(G \times K_m)/\alpha(G \times K_m)$, and since $n(G \times K_m) = n(G)m$, we get $\alpha(G \times K_m) \geq n(G)m/\chi(G) \geq n(G)$.

Now suppose that $\alpha(G \times K_m) \geq n(G)$, and let S be a maximum size independent of $G \times K_m$. Let the vertices of K_m be labeled $y_1, \ldots, y_m$, and for each $i \in [m]$, let S_i be the vertices of S with second coordinate equal to y_i. Then S_i has the form $\{(x, y_i) : x \in X_i\}$ for some set $X_i \subseteq V(G)$. Since S_i is an independent set, X_i must be an independent set.

For distinct $i, j \in [m]$, y_i and y_j are adjacent in K_m, so S has no pair of vertices of the form $(x, y_i), (x, y_j)$; hence $X_i \cap X_j = \emptyset$. Since $X_1, \ldots, X_m$ are pairwise disjoint subsets of G whose sizes sum to $\alpha(G \times K_m) \geq n(G)$, they partition $V(G)$. Since each is an independent set, they can be used as color classes of an m-coloring of G.

5.1.48 By Brooks's Theorem, each component of G is 3-colorable; hence G is 3-colorable. Let f be a proper 3-coloring, and let S_1, S_2, S_3 be its color classes (independent sets). We may assume that S_1 is smallest. Pick a vertex $v \in S_1$; if it has no edges to S_2 or S_3, move it to that set; if v has exactly one edge e to S_2 or S_3, delete e and then move v to that set. One of those cases must apply, since $d(v) \leq 3$. After the move, each S_i is still an independent set, and S_1 is smaller, so this can be repeated until S_1 is empty. At the end, there are at least $n - |S_1| \geq n - m/3$ edges left, and the graph is bipartite with partite sets S_2, S_3.

5.2.7 If not, then there is some color i, such that for each vertex v of color i, there is some color c_v that does not appear on any neighbor of v. Let S be the set of vertices of color i. Recolor each $v \in S$ by the color c_v. Since S is an independent set, for each $v \in S$, no vertex of $N(v)$ is recolored; therefore the new coloring is still a proper coloring. It uses less than k colors, a contradication.

5.2.9 Let G' be the graph created from G, with U and w as specified in class and in the book, and let $k = \chi(G)$. The proof of Mycelski shows that $\chi(G') = k + 1$, so we need to show that for any $e \in E(G')$, $G' - e$ is k-colorable. Sketch:

If $e \in E(G)$, we can properly color $G - e$ with colors $[k-1]$, give color k to all of U, and use any color from $[k-1]$ for w.

If $e = u_i w$, we can k-color G such that v_i is the only vertex with color 1, copy the colors from $V(G)$ to the corresponding vertices in U, then color w with color 1.

Otherwise $e = v_i u_j$ where $v_i v_j \in E(G)$. Then we do a $(k-1)$-coloring on $G - v_i v_j$, and copy the colors to corresponding vertices in U. This is a proper coloring except for the edges $v_i v_j$ and $v_j u_i$, so we recolor v_j with a new color, and use that color for w, too.

5.2.15 Let G be a triangle-free graph. If $\Delta(G) < 2\sqrt{n}$, then we can apply Brooks's Theorem unless G is K_2 or an odd cycle, and in each case we have $\chi(G) \leq 2\sqrt{n}$. So we may assume that $\Delta(G) \geq 2\sqrt{n}$, and let v be a vertex with $d(v) \geq 2\sqrt{n}$. $N(v)$ is an independent set and $G - N(v)$ is triangle free, so we can apply induction to color with at most $1 + 2\sqrt{n - |N(v)|}$ colors. $2\sqrt{n - 2\sqrt{n}} \leq 2\sqrt{n} - 1$ is true (square both sides), so this suffices.

Alternatively: Repeatedly choose a vertex v of maximum degree, give $N(x)$ a single color, then delete $N(x)$; repeat this $\lfloor\sqrt{n}\rfloor$ times, and let H be the subgraph that remains. (Each $N(x)$ is an independent set since G is triangle-free.)

For each of the first $\lfloor\sqrt{n}\rfloor$ steps, we pick a vertex x of degree at least $\Delta(H)$. Therefore $n(H) \leq n - \Delta(H)\lfloor\sqrt{n}\rfloor > n - \Delta(H)(\sqrt{n} - 1)$. If $\Delta(H) \geq \sqrt{n}$ then $n(H) \leq \sqrt{n}$, so H is $\lfloor\sqrt{n}\rfloor$-colorable. H is also $\lfloor\sqrt{n}\rfloor$-colorable if $\Delta(H) \leq \lfloor\sqrt{n}\rfloor - 1$ by greedy coloring, or if $\Delta(H) = \lfloor\sqrt{n}\rfloor$ and Brooks's Theorem applies. A proper $\lfloor\sqrt{n}\rfloor$-coloring of H suffices, so we may assume that $\Delta(H) = \lfloor\sqrt{n}\rfloor$ and H is an odd cycle or $n(H) \leq 2$. So $\Delta(H) \leq 2$, so $n < 9$, and these cases can be handled somehow...

5.2.25 (a) Suppose that G is simple and $\sum_{v \in V(G)} \binom{d(v)}{2} > (m-1)\binom{n}{2}$. The first quantity is the number of subgraphs isomorphic to P_3 (counted according to their center vertices). By counting P_3-subgraphs according to their pairs of endpoints, this must equal the sum, over all pairs of distinct vertices $\{u, v\}$, of the number of common neighbors of u and v. If G does not contain $K_{m,2}$ as a subgraph, then every pair of vertices has at most $m - 1$ common neighbors. so the latter sum is at most $(m-1)\binom{n}{2}$.

(b) Note that $2\sum_{v \in V(G)} \binom{d(v)}{2} = \sum_{v \in V(G)} d(v)^2 - \sum_{v \in V(G)} d(v)$. Balancing the degrees (by subtracting ϵ from $d(u)$ and adding ϵ to $d(v)$ when $d(u) < d(v)$) can only decrease the sum of their squares, so $\sum_{v \in V(G)} d(v)^2 \geq \sum_{v \in V(G)} (2e/n)^2$, which equals $n(4e^2/n^2) = 4e^2/n$. Also $\sum_{v \in V(G)} d(v) = 2e$. Therefore $\sum_{v \in V(G)} \binom{d(v)}{2} \geq \frac{1}{2}(4e^2/n - 2e) = e(2e/n - 1)$.

(c) If $e \geq \frac{1}{2}(m-1)^{1/2}n^{3/2} + n/4$, then $2e/n \geq (m-1)^{1/2}n^{1/2} + 1/2$ and $2e/n - 1 = (m-1)^{1/2}n^{1/2} - 1/2$, so their product is $(m-1)n - 1/4$. Therefore $e(2e/n - 1) \geq (m-1)n^2/2 - n/8$, which is more than $(m-1)\binom{n}{2}$ if and only if $-n/8 > (m-1)(-n/2)$, or $m - 1 > 1/4$, which is true if $m \geq 2$. Then parts (a) and (b) finish the argument. If $m = 1$, then $\frac{1}{2}(m-1)^{1/2}n^{3/2} + n/4 = n/4$, and having more than $n/4$ edges isn't even enough to guarantee a matching, let alone a copy of $K_{1,2}$; that is, the statement is false for $m = 1$.

(d) Fix any n points in the plane, and let G be the graph where points are vertices and two vertices are adjacent when their distance in the plane is exactly 1. Note that two vertices can have at most two common neighbors. Then apply part (c) with $m = 3$.