Math 453, Spring 2008, Instructor: Pelsmajer
Assigned on March 11.

(A) Let a_n be the number of ways to put n books into some stacks, say, in identical cardboard boxes. No empty boxes, all books distributed, and books are distinguishable—say, labeled $1, \ldots, n$. (We already know that if we separate into exactly k stacks, the answer is $\binom{n-1}{k-1}n!/k!$. But here the number of boxes are restricted.)

Find a simple exponential generating function for $\{a_n\}_{n=0}^{\infty}$, and then use the "$x(d/dx)\log$ operation" method, as seen on p23-24 of Wilf (online edition), to get a three-term recurrance relation.

Do #3, 6, & 7 from p104 of Wilf. (Just in case, I'll tell you that if a graph has all vertices of degree at most 2, then each component is either a path or a cycle.)

Also, read section 7.1.1, then Examples 7.3 and 7.4.

Assigned on March 14.

For all these problems, the intention is that you use Inclusion-Exclusion. However, it wouldn't surprise me if you can use other techniques to solve these problems at this point.

If you come up with an another, significantly different (in my opinion) way to solve it, I'll give you some bonus points. But I want to see a P.I.E. solution for each problem.

Finally, note that: Although we discussed "exactly a set T of properties hold", and "exactly t properties hold" during class, for the problems below we only need the case $T = \emptyset$, or $t = 0$; this mostly corresponds to the material in section 7.1 of Roberts and Tesman.

From section 7.1 of Roberts and Tesman, do #11, 28, 29, 30.

(B) How many permutations of $[n]$ leave no odd number fixed?

(C) How many words are there with exactly two copies of each number from 1 to n, such that no two consecutive letters are the same? (For example, for $n = 3$, 323121 is okay, but 322131 is not.)

(D) Consider a set of n boys and n girls. We pair up the $2n$ people as lab partners (i.e., we partition them into n sets of size 2), but with the following restrictions.

(D.1) For each i, the ith tallest boy is not paired with the ith tallest girl.

(D.2) Same as (D.1), but also each pair has one person of each sex.

(Bonus/E) How many permutations of $[n]$ have no consecutive pair of numbers that differ by exactly 1?

(For example, 65473128 is not okay for three reasons: 65, 54, 12.)

(The solution will be a double sum.)

(Bonus/F) Modifying the technique from class, prove the generalized P.I.E. Theorem:

If A is a set of objects, and P is a set of properties that the objects can have, and T is a particular subset of properties, then

$$e(T) = \sum_{S\,:\,T\subseteq S\subseteq P} (-1)^{|S|-|T|} N(S).$$

Here, $e(T)$ is the number of objects that have exactly the set of properties T, and no other properties of P. $N(S)$ is the number of objects that have all the properties in S (and perhaps other properties as well).