

Solutions to Assignment #8

3.3.1 Let S be the set of vertices of degree 3. Then $o(G - S) = 4$, so by Tutte's Theorem, the answer is no.

3.3.2 If we delete the four "rectangle corner" vertices, we obtain six odd components. Thus, the maximum of $o(G - S) - |S|$ taken over $S \subseteq V(G)$ is at least 2. According to the Berge-Tutte formula, a maximum matching saturates at most $n(G) - 2$ vertices.

It's not hard to find such a matching (do it!).

3.3.8 If G decomposes into zero 1-factors, then every vertex of G is an isolated vertex, so none are cut-vertices. Thus, we may assume that G has a 1-factor. Then $n(G)$ is even. Then $n(G - x)$ is odd for any $x \in V(G)$. Then $G - x$ must have an odd component H . A 1-factor in G must have an edge from H to x , and no other edges incident to x . Since G decomposes into 1-factors, $N(x) \subseteq V(H)$. If x were a cut-vertex then it would be adjacent to more than one component of $G - x$, so x is not a cut-vertex.

Replace one edge xy of K_4 by a path x, z, y where $z \notin V(K_4)$. Make two copies of this graph and add an edge between the two vertices labeled z . We get a 3-regular graph G in which the vertices labeled z are both cut-vertices, and it's easy to find a perfect matching in G .

3.3.11 Suppose that G has a matching M that saturates $T \subseteq V(G)$. Let G_M be G restricted to the vertices that are incident to M ; then G contains T . Consider any $S \subseteq V(G)$ and let H be an odd component of $G - S$ contained in G_M . Then there must be an edge $e \in M$ such that H contains exactly one endpoint of e . The other endpoint of e must be in S . Since distinct edges of M have distinct endpoints in S , the number of components like H is at most $|S|$. Anything contained in $G[T]$ is also contained in G_M , so the number of odd components of $G - S$ contained in $G[T]$ is at most $|S|$.

For the other direction, we are given G and $T \subseteq V(G)$, and we assume that for any $S \subseteq V(G)$, the number of odd components of $G - S$ that are contained in $G[T]$ is at most $|S|$. We need to find a T -saturating matching in G . Let $G' = G \cup ((G - T) \vee K_n)$, with $n = n(G)$. A perfect matching in G' would contain a T -saturating matching M , and since any edge of $E(G')$ incident to T is in G , M would be the desired matching. We will show that G' has a perfect matching using a proof by contradiction and Remark 3.3.5: then, since $n(G') = 2n$ (an even number), we may assume that $o(G' - S) \geq |S| + 2$ for some $S \subseteq V(G')$.

If S contains $V(K_n)$, then $|S| \geq n \geq o(G - S) = o(G' - S)$, a contradiction. So assume that S does not contain $V(K_n)$. In $G' - S$, $V(K_n) - S$ is in the same component as $G' - S \cup T$, and the other components of $G' - S$ are contained in $G[T]$. The latter are also components of $G - S \cap V(G)$, so by the original assumption, at most $|S \cap V(G)|$ of these have odd order. Thus there are at most $|S \cap V(G)| + 1$ odd components of $G' - S$, which is at most $|S| + 1$, a contradiction.

3.3.15 Let G be a 3-regular graph with at most 2 cut-edges. By Corollary 1.3.6, $n(G)$ is even, so we can apply Remark 3.3.5: If G has no 1-factor, then $o(G - S) \geq |S| + 2$ for some $S \subseteq V(G)$. Fix such an S . (The rest is similar to the proof of Corollary 3.3.8, p139.)

Consider an odd component H of $G - S$. The number of edges from H to S is $\sum_{v \in V(H)} d_G(v) - 2e(H)$, which equals $3n(H) - 2e(H)$, which is odd since $n(H)$ is odd; hence it is at least three unless H is connected to S by a single edge, which would be a cut-edge of G . Thus, the number of edges between S and the odd components of $G - S$ is at least $3[o(G - S) - 2] + 2$. Since G is 3-regular the total number of edges incident to S is at most $3|S|$, so $3|S| \geq 3[o(G - S) - 2] + 2$. But then $o(G - S) \leq |S| + 4/3$, a contradiction.

3.3.20 Let G be a 3-regular simple graph with a perfect matching M . Then $G - M$ is 2-regular, which means that it is the disjoint union of some cycles. For each cycle C in $G - M$, pick a direction around the cycle, and associate each vertex on C with the edge that follows it. For each edge $uv \in M$, u and v are associated with edges on cycles in $G - M$; these three edges form a copy of P_4 . It's not hard to verify that each edge of G is in exactly one of these P_4 s.

Let G be a graph that decomposes into copies of P_4 . Let M be the set of the middle edges of these P_4 s. Each edge of M is in only one P_4 . Suppose that edges $e, f \in M$ share an endpoint v . Each copy of P_4 containing v has two edges incident to v . Since e and f are in edge-disjoint P_4 s of the decomposition, v has degree at least 4, a contradiction.