Solutions to Assignment #9, version 2.

4.1.9 Take the union of two copies of K_{m+1} so that they intersect on a set A of exactly k vertices. Then take the disjoint union with another K_{m+1}, and add a perfect matching M of size ℓ from it to one of the first two complete graphs. Call this graph G.

Deleting a set of size less than k will not disconnect any K_{m+1}, and will not eliminate A, nor M, so the graph would remain connected. Therefore $\kappa(G) \geq k$, and since A is a separating set of size k, $\kappa(G) = k$. Arguing $\kappa'(G) = \ell$ is similar. Every vertex is in a copy of K_{m+1} so every vertex has degree at least m. Since $m + 1 \geq \ell$ there is a vertex in the last copy of K_{m+1} with degree equal to m, and thus $\delta(G) = m$.

4.1.15 The Petersen graph G is 3-regular, so Theorem 4.1.11 gives $\kappa(G) = \kappa'(G)$; thus it suffices to show that $\kappa'(G) \geq 3$. By Remark 4.1.8, there is a set S with $|[S, \overline{S}]| = \kappa'(G)$, and by symmetry we can assume that $|S| \leq |\overline{S}|$, so $|S| \leq 5$. For a contradiction, assume that $|[S, \overline{S}]| \leq 2$. Then $|S| > 3$ by Corollary 4.1.13.

By Proposition 4.1.12, $|[S, \overline{S}]| = 3|S| - 2e(G[S])$. If $|S| = 4$ then $e(G[S]) \geq 5$ so $G[S]$ contains a cycle C, but the girth is 5 so we have a contradiction. Otherwise $|S| = 5$, so $e(G[S]) \geq \lceil 13/2 \rceil = 7$. So $G[S]$ contains a cycle C, which must be a 5-cycle because of the girth; and another edge will (with C) form a smaller cycle, contradicting the girth.

4.1.25 (a) Suppose that $v \in S$ has no neighbor in $\overline{S}$. Then $N(v) \subseteq S - v$, so $|N(v)| < |S|$. Since the diameter of G is 2, every vertex of $\overline{S}$ is connected to $N(v)$ by an edge; therefore $|[S, \overline{S}]| \geq |\overline{S}|$. But $|\overline{S}| \geq |S| > |N(v)| \geq \delta(G) \geq \kappa'(G) = |[S, \overline{S}]|$, so we have a contradiction.

(b) By part (a), $|[S, \overline{S}]| \geq |S|$, and for a contradiction, with Corollary 4.1.13, we may assume that $|S| > \delta(G)$. But $|[S, \overline{S}]| = \kappa'(G) \leq \delta(G)$, so we have a contradiction.

4.1.28 Consider edge cuts $F = [S, \overline{S}]$ and $F' = [T, \overline{T}]$, and let $A = S \cap T$, $B = S \cap \overline{T}$, $C = T \cap \overline{S}$, and $D = \overline{S} \cap \overline{T}$. The intersection of F and F' is the union of $[A, D]$ and $[B, C]$, so the symmetric difference of F and F' is the union of $[A, B]$, $[A, C]$, $[B, D]$, and $[C, D]$. And this equals $[A \cup D, B \cup C]$.

4.1.32 If every block is Eulerian, then every vertex has even degree within each of its blocks. Since the graph decomposes into blocks, and the sum of even numbers is even, every vertex has even degree in G.

Suppose that every vertex of G has even degree. Any component of G is Eulerian, so we may assume that G has a component H that is not a block. Then the block-cutpoint graph of H has a leaf B that is a block in H, with neighbor v that is the cut-vertex of H contained in B. Let $G' = G - [V(B) - v]$. By induction (on the number of blocks), each block of G' is Eulerian, and this includes all the blocks of G except for B. All vertex degrees in B are even except possibly $d_B(v)$. By the degree-sum formula $2e(B) = \sum_{x \in V(B)} d_B(x)$, $d_B(v)$ must be even as well. Hence, B is Eulerian.