Solutions to Assignment #10, Dec 10 version.

**4.2.12** $\kappa(G) \le \kappa'(G) \le 3$ by Theorem 4.1.9, so we need only show that $\kappa(G) \ge \kappa'(G)$. Let $k = \kappa'(G)$. Between each pair of vertices $u, v$, there is a set of $k$ edge-disjoint paths. If two of these paths intersected at a vertex $z \notin \{u, v\}$, then $z$ would have degree at least 4 because each path contributes two edges incident to $z$. So these $k$ paths must actually be internally-disjoint! Then by Theorem 4.2.21, $\kappa(G) \ge k$.

*Alternatively:* $\kappa(G) \le \kappa'(G) \le 3$ by Theorem 4.1.9, so we need only show that $\kappa(G) \ge \kappa'(G)$. If $G$ is a complete graph this is true, so assume that $G$ is not complete. Then we may let $S$ be a minimum separating set; note that $\kappa(G) = |S|$. Let $x, y$ be vertices in different components of $G - S$. Let $\mathcal{P}$ be a maximum-size set of pairwise edge-disjoint $x, y$-paths. By Menger's Theorem, $\kappa'(G) \le |\mathcal{P}|$. Two paths in $\mathcal{P}$ cannot share a vertex $z \in S$, since $d(z) \le 3$ and they are edge-disjoint. Therefore $|\mathcal{P}| \le |S|$.

**4.2.23** Let $G$ be an $X, Y$-bigraph. The easy direction is $\alpha'(G) \le \beta(G)$; we need to show that $\alpha'(G) \ge \beta(G)$, i.e., that there exists a matching in $G$ of size at least $\beta(G)$.

Modify $G$ by adding vertices $s, t$ with $N(s) = X$ and $N(t) = Y$; call the new graph $H$. Note that an $s, t$-separating set in $H$ is a vertex cover in $G$ and vice-versa; therefore $\kappa_H(s, t) = \beta(G)$. By Menger's Theorem, there is a set $\mathcal{P}$ of $\kappa_H(s, t)$ internally-disjoint $s, t$-paths in $H$. Each path $P$ contains an edge $e_P$ in $G$, and these edges form a matching in $G$, as desired.

**4.2.24** Given a $k$-connected graph $G$ with disjoint subsets $S, T$ each of size at least $k$, add vertices $s, t$ with $N(s) = S$ and $N(t) = T$; call the new graph $H$. There can be no $s, t$-cut $X$ of size less than $k$: Then $S - X$ and $T - X$ would both be nonempty, with $S - X$ in the same component of $H - X$ as $s$, and $T - X$ and $t$ in another component of $H - X$. Since $G - X = H - X - \{s, t\}$, $S - X$ and $T - X$ are in different components of $G - X$, showing that $G - X$ is disconnected, which contradicts the $k$-connectedness of $G$. Then Menger's Theorem yields $k$ pairwise internally-disjoint $s, t$-paths in $H$. Remove $s$ and $t$ from each path. (If you prefer paths that only intersect $S \cup T$ at its endpoints, take minimal subpaths from $S$ to $T$.)

**4.3.8** Suppose that $w(v)$ is the desired weight for each vertex $v \in V(G) - \{x, y\}$; let $w(x)$ and $w(y)$ be infinite. We create a new network $G'$ as follows: For each vertex $v \in V(G)$, create vertices $v^-, v^+$ and and edge $v^-v^+$ of capacity $w(v)$. For each directed edge $uv$, let $u^+v^-$ be an edge of infinite capacity. Let $s = x^-$ and $t = y^+$. A flow in $G'$ corresponds to a flow in $G$ with the amount of flow through each vertex $v \notin \{x, y\}$ bounded by $w(v)$. *Alternatively:* For each edge $uv$, let $c(uv)$ equal the minimum of the capacities of its endpoints, and let $s = x$ and $t = y$. Thus we have a network $G'$, and a flow in $G'$ is precisely a valid flow in $G$.

If $xy \in E(G)$ then there is no maximum flow, because $xy$ can carry any amount of flow. Otherwise, the Max Flow Min Cut Theorem applies, and the Ford-Fulkerson labelling algorithm can be used to find the optimum flow.

**4.3.13** (a) Make a network with a vertex $x_i$ for the $i$th company and a vertex $y_j$ for the $j$th group. Assign edges with weights: $w(sx_i) = m_i$ for each $i$, $w(x_iy_j) = 1$ for each $i, j$, and $w(y_jt) = n_j$ for each $j$. The integrality theorem applies, so there is a flow of value $\sum m_i$ if and only if there are unit flows of that value that can be combined to form a flow of that value. The latter corresponds to a way to assign all representatives to groups (as desired). So we run Ford-Fulkerson and see whether there is a flow of value at least (actually it would have to be equal to) $\sum m_i$.

(b) By part (a) and the Max-flow Min-cut Theorem, there is a good assignment if and only if the minimum capacity of a source/sink cut is at least $\sum m_i$. Let $[S, T]$ be a minimum-capacity source/sink cut. Let $k = |S \cap \{x_i\}|$ and $\ell = |S \cap \{y_j\}|$. The capacity of $[S, T]$ is the sum of (i) $\sum_{x_i \in T} m_i$, (ii) $|S \cap \{x_i\}| \cdot |T \cap \{y_j\}|$, and (iii) $\sum_{y_j \in S} n_j$. If we replace changed $S \cap \{x_i\}$ and $T \cap \{x_i\}$ to be $\{x_1, \ldots, x_k\}$ and $\{x_{k+1}, \ldots, x_p\}$, respectively, (i) would not increase since $m_1 \ge \cdots \ge m_p$, and (ii) and (iii) would be unaffected, so we would obtain a new minimum-weight source/sink cut. Thus we may assume that $S \cap \{x_i\} = \{x_1, \ldots, x_k\}$ and $T \cap \{x_i\} = \{x_{k+1}, \ldots, x_p\}$. Similarly, we may assume that $S \cap \{y_i\} = \{y_1, \ldots, y_{q-\ell}\}$ and

$T \cap \{y_i\} = \{y_{q-\ell+1}, \ldots, y_q\}$. Thus the capacity of $[S, T]$ is $\sum_{i=k+1}^{p} m_i + k(q-\ell) + \sum_{j=1}^{\ell} n_j$. This is at least $\sum m_i = \sum_{i=1}^{p} m_i$ if and only if $k(q-\ell) + \sum_{j=1}^{\ell} n_j \geq \sum_{i=1}^{p} m_i - \sum_{i=k+1}^{p} m_i$, which equals $\sum_{i=1}^{k} m_i$.