Solutions to Assignment #7, version 11/18/08

3.1.9 Let M be a maximal matching in G, and let M' be a maximum matching in G. Each edge $e \in M'$ must share an endpoint with some edge of M, since otherwise $M \cup \{e\}$ would be a matching, contradicting the maximality of M. Since two edges of M' do not share any endpoints, this means that the edges of M must be have at least $|M'|$ distinct endpoints. Since M has $2|M|$ endpoints and $|M'| = \alpha'(G)$, we have $2|M| \geq \alpha'(G)$.

Alternatively, apply Lemma 3.1.9: each component H of the graph $(V(G), M \triangle M')$ is an alternating path or even cycle. None of these paths consists of a single edge in M' (or in M), since then M (M') would not be maximal; hence $e(H) \geq 2$. If $e(H)$ is even then $|E(H) \cap M| = |E(H) \cap M'|$. If $e(H)$ is odd, $|E(H) \cap M|/|E(H) \cap M'| \geq [\frac{1}{2}(e(H) - 1)]/[\frac{1}{2}(e(H) + 1)] \geq (3 - 1)/(3 + 1) = 1/2$.

3.1.10 By Lemma 3.1.9, each component of a graph with edge set $M \triangle N$ is a path or an even cycle. Since $|M| > |N|$, some component H has more edges from M than N; clearly H must be a path alternating edges of M and N, with first and last edges in M. Let $M' = M \triangle E(H)$ and $N' = N \triangle E(H)$. Since this only swaps edges from $M - N$ and $N - M$, $M' \cup N' = M \cup N$ and $M' \cap N' = M \cap N$. Since $|M \cap E(H)| = |N \cap E(H)| + 1$, $|M'| = |M| - 1$ and $|N'| = |N| + 1$. It remains to prove that M' and N' are matchings; we show that every vertex is incident to at most one edge of M' and at most one of N'. By the construction, we need only consider vertices of the path H. Since each interior vertex of H is incident to exactly one edge in M and N each, both in $E(H)$, it is incident to exactly one edge of M' and exactly one edge of N'. An endpoint v of H is incident to an edge $e \in M \cap E(H)$, so v is not incident to any other edge of M, including all of $M \cap N$. By choice of H, v is not incident to any other edge of $M \triangle N$, either. Since $(M \cap N) \cup (M \triangle N) = M \cup N = M' \cup N'$, v is incident to exactly one edge of $M' \cup N'$. It follows that M' and N' are matchings.

3.1.21 For any edge $e \in E(G)$, it has endpoints $x \in X$ and $y \in Y$. Let $H = G - \{x, y\}$. For any $S \subseteq X - x$, $N_H(S) = N_G(S) - y$, so $|N_H(S)| \geq |N_G(S)| - 1$, which by assumption is at least $|S|$. Hall's Theorem gives a perfect matching of H; add e for a perfect matching of G.

3.1.29 *Part 1:* Each vertex in a vertex cover S of G is incident to at most $\Delta(G)$ edges, so $e(G) \leq |S|\Delta(G)$. Therefore $\beta(G) \geq e(G)/\Delta(G)$. Now apply the König-Egerváry Theorem: there is a matching of size at least $e(G)/\Delta(G)$. *Part II:* Suppose that G is a subgraph of $K_{n,n}$ with $e(G) > (k-1)n$. Then $\Delta(G) \leq n$, so $e(G)/\Delta(G) > k - 1$. By Part I, G has a matching of size greater than $k - 1$, i.e., size at least k.

3.1.31 Let G be an X, Y-bigraph, and suppose that for any $S \subseteq X$, $|N(S)| \geq |S|$. Let T be a minimum vertex cover of G. Since edges incident to $X - T$ are not incident to $X \cap T$, they must all be incident to $Y \cap T$. Hence, $N(X - T) \subseteq Y \cap T$. Then $|X - T| \leq |N(X - T)| \leq |Y \cap T|$, so $|X| = |X \cap T| + |X - T| \leq |X \cap T| + |Y \cap T| = |T|$. Now apply the König-Egerváry Theorem: there is a matching of size at least $|T| \geq |X|$, so it must saturate X (and have size exactly equal to $|X|$).