Math 454/553, Fall 2008, Instructor: Pelsmajer

Solutions to Assignment #5, version 2.

2.1.24 A nontrivial tree has a leaf v; let u be the neighbor of v. Since $\{v\}$ is an independent set, there is a maximal independent set S containing v. Similarly, there is a maximal independent set S' containing u. Since u and v are neighbors, $u \notin S$, and thus $S \neq S'$.

Alternatively: A tree is bipartite (an acyclic graph has no odd cycles), so we may let A, B be partite sets. Each is an independent set, so we can extend them to maximal independent sets; call these A', B', with $A \subseteq A'$ and $B \subseteq B'$. If $A' = B'$ then this set contains $A \cup B$, so it equals $V(G)$. Then G has no edges, and the only edgeless tree is trivial, a contradiction.

A tree T is a star iff its diameter is at most 2. Since a nontrivial star is isomorphic to the complete bipartite graph $K_{1,n-1}$ for some $n \geq 1$, it has exactly two maximal independent sets (its partite sets).

If T is a tree of diameter at least 3, T contains two vertices u, v at distance at least 3. Let P be a shortest u,v-path, and let u, x, y, z be the first four vertices along P (so z might equal v). The sets $\{u, y\}$, $\{x, z\}$, and $\{u, z\}$ are all independent sets, since otherwise there would be an edge that allowed us to modify P and get a shorter u,v-path. Each of the three sets extends to a maximal independent set in T; say S, S', S''. If $S = S''$ then S would contain $\{u, y, z\}$, but yz is an edge. Therefore $S \neq S''$, and similarly $S \neq S'$ and $S' \neq S''$.

2.1.26 For every vertex v, $G - v$ is an $(n-1)$-vertex tree, so $e(G - v) = n - 2$. Also, $e(G - v) = e(G) - d(v)$, so $d(v) = e(G) - n + 2$ which means that G is regular. Therefore $d(v) = 2e(G)/n$ for all v, and $2e(G)/n = e(G) - n + 2$ which simplifies to $e(G) = n$.

Note that G is 2-regular. There cannot be a loop at any vertex v, since then $G - u$ would not be acylic for $u \neq v$. Since $G - v$ is connected and v has degree 2 but is not incident to a loop, G is connected. A 2-regular connected graph is a cycle, so $G \cong C_n$.

2.1.29 Let T be an n-vertex tree. A tree is acyclic so it has no odd cycles; therefore T is bipartite.

If $n = 1$, then T has no leaves, so the desired result is FALSE! Assume that $n \geq 2$. Let X and Y be a bipartition of T such that $|X| \geq |Y|$. Since $|X| + |Y| = n$, $|X| \geq \lceil n/2 \rceil$. Suppose that X has no leaves.

Version 1 (proof by counting): Since X has no leaves and $n \neq 1$, each vertex of X has degree at least 2. Every edge has exactly one endpoint in X, so $e(T) = \sum_{x \in X} d(x)$. But $\sum_{x \in X} d(x) \geq 2|X| \geq 2\lceil n/2 \rceil \geq n$, so $e(T) \geq n$ which contradicts Theorem 2.1.4.

Version 2 (extremal proof): Suppose that T is a minimum counterexample; i.e., T is a minimum-size tree with bipartition X, Y with $|X| \geq |Y|$ such that X has no leaves. By Lemma 2.1.3, since $n \geq 2$, T has a leaf v with neighbor u. Then $v \in Y$, and $u \in X$.

$T - v$ is a tree with bipartition $(X, Y - v)$. Since $|X| > |Y - v|$ and $T - v$ is a smaller tree, $T - v$ must have a leaf in X. Since $d_{T-v}(x) = d_T(x)$ for all $x \in X - u$, and T has no leaves in X, it follows that u must be the leaf of $T - v$ in X. Then $T - \{u, v\}$ is a tree, with partite sets $X - u$ and $Y - v$. Every vertex in $X - u$ has the same degree in $T - \{u, v\}$ as in $T - v$, so $T - \{x, u\}$ has no leaves in $X - u$. $|X - v| \geq |Y - u|$, so we contradict the choice of T as a minimum-size counterexample.

Note: Version 2 could be rephrased as a proof by induction.

2.1.36 First, note/recall that the symmetric difference $S \triangle S'$ of two finite sets S, S' has even size if $|S|$ and $|S'|$ are both even, or if they are both odd; $|S \triangle S'|$ is odd if one of $|S|, |S'|$ is odd and the other is even. Therefore, if H and H' are subgraphs of a tree T, and F is the spanning subgraph of T with $E(F) = E(H) \triangle E(H')$, then $d_F(v)$ is odd iff v has odd degree in exactly one of H, H'.

Suppose that H, H' are distinct spanning subgraphs of T in which every vertex has odd degree, and let F be the spanning subgraph of T with $E(F) = E(H) \triangle E(H')$. Then F is an even graph, and $E(F) \neq \emptyset$. But F has a component with at least one edge, and this is a tree with at least two leaves, which have odd degree; contradiction.

For the other direction, we construct a good spanning subgraph H of any tree T.

Method 1: Partition $V(T)$ into pairs. Each pair is joined by a unique path in T. Let H be the spanning subgraph with $E(H)$ equal to the symmetric difference of the edge-sets of these paths, i.e., $E(H) = (\cdots((E(P_1) \triangle E(P_2)) \triangle E(P_3)) \triangle \cdots \triangle E(P_k))$. Since every vertex v of H appears as an endpoint of exactly one path, v will have odd degree in H.

Method 2 (Induction): Basis case: If $n(T) = 2$ then let $H = T$. So we may assume that $n(T) \geq 4$.

If T has two leaves u, v with a common neighbor w, then apply induction to $T - \{u, v\}$ to get a good spanning subgraph F of $T - \{u, v\}$. Add the edges uw, vw to $E(F)$ and add u, v to $V(F)$ to get the desired subgraph of T.

If not, then let T' be obtained by deleting all leaves of T. T cannot be a star (since then either two leaves have a common neighbor, or $n(T) \leq 2$) so $n(T') \geq 2$. Let u be a leaf in T'. By the construction of T', u is not a leaf in T, so $d_T(u) \geq 2$. Since no vertex is adjacent to two leafs in T, u is adjacent to exactly one leaf v in T, and $d_T(u) = 2$. Apply induction to $T - \{u, v\}$ to get a good spanning subgraph F of $T - \{u, v\}$. Add the edge uv to $E(F)$ and add u, v to $V(F)$ to get the desired subgraph of T.

2.1.47.a. Consider any vertices u, v, w. If the component containing v does not contain both u and w, then $d(u, v) + d(v, w)$ is infinite and the desired inequality is true. Otherwise, u, v, and w are all in one component. Let P be a u, v-path of minimum and let Q be a v, w-path of minimum length. Then the concatination of P and Q forms a u, w-walk of length $d(u, v) + d(v, w)$. Since this u, w-walk must contain a u, w-path P and any u, w-path has length at least $d(u, w)$, we obtain $d(u, v) + d(v, w) \geq \text{length}(P) \geq d(u, w)$.

b. Let u, v be vertices such that $d(u, v)$ is maximized, and let w be a vertex of minimum eccentricity; then $d(u, v) = \text{diam}(G)$ and $\epsilon(w) = \text{rad}(G)$. The latter implies that $d(u, v)$ and $d(v, w)$ are each at most $\text{rad}(G)$. Then $d(u, v) \leq d(u, w) + d(v, w) \leq 2\text{rad}(G)$.

c. Suppose that d, r are positive integers such that $r \leq d \leq 2r$. Add an edge from an endpoint of P_{d-r} to a vertex v of C_{2r} to create a graph G. Since opposite vertices in C_{2r} are at distance r in G, their eccentricities are at least r. Since vertices of P_{d-r} are at distance at least r from the vertex opposite v on C_{2r}, the minimum eccentricity in G is at least r. Since $d - r \leq r$, $\epsilon(v) = r$. Thus, $\text{rad}(G) = r$. Since any two vertices on C_{2r} have distance at most r, and since the maximum eccentricity of a vertex of P_{d-r} in G is clearly $d - r + r = d$, the diameter of G is indeed d.

Alternative: Let u, v be vertices, and take the union of two u, v-paths of length d and one u, v-path of length $2r - d$, such that two paths intersect only at their endpoints. This graph contains an induced r-cycle so its radius is at least r, and since $\epsilon(x) = r$, the radius equals r. It contains an induced $2d$-cycle so the diameter is at at least d. Any two vertices are connected by a path of length at most $\max\{\lfloor \frac{d}{2} \rfloor + \lfloor \frac{2r-d}{2} \rfloor, 2\lfloor \frac{d}{2} \rfloor\}$, which is at most $2\lfloor \frac{d}{2} \rfloor \leq d$, the diameter is exactly d.