Solutions to Assignment #2.

1.2.10 and **1.2.11**

1.2.10a Proof: Let G be a bipartite graph with partite sets X, Y. Any trail alternates between X and Y, so a closed trail must have an even number of edges. An Eulerian circuit is a closed trail that uses every edge of G exactly once, so if an Eulerian circuit exists, then $|E(G)|$ must be even.

1.2.10b Counterexample: $C_3 \cup C_4$, where the cycles intersect at a single vertex.

1.2.11 Counterexample: The union of two cycles at a single vertex v, where e and f are in the first cycle, such that both are incident to v.

1.2.15 Starting at the first endpoint u of W, follow the sequence W. As long as no vertex is repeated, the vertices and edges of the walk form a path. Since W is closed and $\ell(W) > 0$, eventually some vertex must repeat. Consider the first repeated vertex, z.

Proof ending #1: There is a z, z-subwalk of length at least 1; apply induction to it.

Proof ending #2: Preceding z on W is a path P from u to some vertex v, followed by an edge e with endpoints v and z. If e is not in P, then the z, v-portion of P and e form a cycle. Therefore e is in P, and since P only has one edge incident to v, e is that edge. Since e is the last edge of P and the first edge in W after P, e is repeated immediately in W.

Alternative Proof: Consider a maximal path $P = v_0, \ldots, v_k$ in W. There are no loops (since they are cycles) and there are edges (since $\ell(W) > 0$), so $\ell(P) = k \geq 1$. Since there are no cycles, each endpoint of P has degree 1 in the graph. If v_k is not the last vertex in W, then the edge before v_k, $v_{k-1}v_k$, is immediately repeated after v_k. If v_0 is not the first vertex in W, similarly v_0v_1 is repeated before and after v_0. Otherwise $P = W$, which a contradiction since W is closed and $\ell(P) \neq 0$.

1.2.22 Suppose that G is connected, and let S, T be a partition of $V(G)$ with $S, T \neq \emptyset$. Then there is a path P from a vertex of S to a vertex of T. Since P begins in S but ends in T, there is some vertex v that is its first vertex in T. The vertex before v must be in S, and P contains an edge between these vertices.

For the other direction, suppose G is not connected. Let H be a component of G. Then $V(H)$ and $V(G) - V(H)$ form a partition of $V(G)$ into two non-empty sets, and there is no edge between them.

Alternatively, for the other direction: Assume that for any partition of $V(G)$ into nonempty sets, there exists an edge with one endpoint in each set. Let u, v be vertices in G. (We must show that G contains a u, v-path.) Let S be the set of all vertices x for which G contains a u, x-path. If $S = V(G)$ then we're done, so we may assume that $\overline{S} = V(G) \setminus S$ is nonempty. Also, $x \in S$ so S is nonempty.

Now $S, \overline{S}$ partitions $V(G)$ into two nonempty sets, so there must be an edge e with endpoint a in S and b in $\overline{S}$. Since $a \in S$, G contains a u, a-path P. Then P, e, b is a u, b-walk, and (by an early lemma) this walk contains a u, b-path. But then b is in S, contradicting $b \in \overline{S} = V(G) \setminus S$.

1.2.23 Let G be a connected simple graph that is not a complete graph. Then $n(G)$ is at least 3.

a. *Proof:* Pick a vertex x, and let S be the set of vertices adjacent to x, and let $T = V(G) \setminus (S \cup \{x\})$. Since G is connected and $n(G) \geq 3$, S is nonempty. If S contains a pair of nonadjacent vertices, then with x they induce a copy of P_3; thus we may assume that S is a clique. If there is an edge uv with $u \in S$ and $v \in T$, then $G[x, u, v] \cong P_3$; thus we may assume that there are no edges between S and T. If $S = V(G) \setminus \{x\}$ then G would be a complete graph, so T must not be empty. But there is no edge from T to S or to x, so G is not connected, a contradiction.

Alternative proof: For any pair of vertices u, v in any graph G, a minimum-length u, v-path P is an induced path, because any edge of $E(G[V(P)]) - E(P)$ can be used to get a shorter u, v-path. (This also works if P is a u, v-path such that $V(P)$ is minimal.) Also, every 3-vertex subpath of P is an induced path.

Thus, for an arbitrary vertex x, if there exists a vertex v that is not adjacent to x, then a shortest x, v-path has length at least 3, and its first three vertices induce a copy of P_3 that contains x. So we may assume that $N(x) = V(G) - \{x\}$. Since G is not complete, there exist two vertices $u, v \in N(x)$ that are not neighbors. Then u, x, v is an induced path in G.

b. *Counterexample:* Let G be the union of C_3 and P_2 with one shared vertex, x. C_3 has one edge e that is not incident to x. There are four vertex subsets of size 3, and none of them induce a copy of P_3 that contains e.

1.2.26 First, let G be a bipartite graph with bipartition X, Y, and let H be a subgraph of G. Then $V(H) \cap X$ and $V(H) \cap Y$ are independent sets that partition $V(H)$, and one must have size at least $|V(H)|/2$.

Now suppose that G is a graph and that every subgraph H of G has an independent set of size at least $|V(H)|/2$. We prove that G is bipartite by contradiction. If G is not bipartite, then G contains an odd cycle, C_k. Any $S \subset V(C)$ with at least $|V(C)|/2$ vertices must actually contain more than half of $V(C)$, since k is odd. Then S must contain two consecutive vertices along the cycle, which are adjacent. Contradiction.

1.2.37 Suppose that u is connected to v and that v is connected to w. That is, G contains a u, v-path P and a v, w-path Q. P intersects Q eventually at v, so we can let z be the *first* vertex along P that is also in Q. Let P' be the u, z-portion of P and let Q' be the z, w-portion of Q. P' doesn't intersect Q except at z, so P' concatinated with Q' has no repeated vertices so it is a path. This u, w-path shows how This path shows how u is connected to w.