Math 454/553, Fall 2008, Instructor: Pelsmajer

Solutions to Assignment #1.

1.1.11 Since the 4 vertices of the middle row has no triangle on them, any clique contains at most 2 of those 4 vertices. Since there are 2 other vertices, any clique has size at most 4. The top, bottom, and 2 center vertices form a clique of size 4, so the maximum size of a clique is 4.

An independent set in a graph is a clique in the complement of the graph. The complement of this graph is two isolated vertices with a copy of P_4, so its maximum clique size is 2. Thus the maximum size of an independent set in the original graph is 2.

1.1.13 Let k be any positive integer. Let A be the set of k-tuples with an odd number of ones, and let B be the set of k-tuples with an even number of ones. If a k-tuple has m ones, then its neighbors have either $m+1$ ones or $m-1$ ones. Hence the neighbors of a vertex in A are in B, and vice-versa. Therefore G is bipartite with partite sets A and B.

1.1.19 Label the graphs F, G, H from left to right.

The middle vertices of H form two copies of C_5. Redraw H so that one of these 5-cycles is much bigger, so that it surrounds the outer 10-cycle in H. This way of redrawing H gives a drawing that is nearly identical to the drawing of G, so one can easily see that $H \cong G$.

To show that F is not isomorphic to G or H, we will show that F contains no 5-cycles (this suffices since G and H do contain 5-cycles). Starting at any vertex x on its inside, label its neighbors y_1, y_2, y_3. These vertices have six total neighbors besides x, and label them $z_1, \ldots, z_6$. Note that there is no edge of the form z_iz_j. Therefore, x is not in a 5-cycle. By symmetry, none of the inner 10 vertices are in a 5-cycle. Thus, if there is a 5-cycle it only uses vertices on the outside ring, and edges between those vertices. But those vertices and edges form a copy of C_{10}, which doesn't contain a 5-cycle; so there is no 5-cycle.

1.1.27 Let x be a vertex in G, and let S be the set of neighbors of x. (S is called *the neighborhood of* x.) G has no 1-cycles or 2-cycles since the girth is 5; therefore G is a simple graph, and $|S| = \deg(x)$.

For each vertex $y \in S$, let S_y be the set of neighbors of y, except for x. *Claim: the sets S_y ($y \in S$) are all disjoint, and do not overlap x or S either.* Then we may conclude that $|V(G)| \geq 1 + |S| + \sum_{y \in S} |S_y|$. Since every vertex has degree at least k, $|S| \geq k$ and $|S_y| \geq k-1$ for all $y \in S$. Then $|V(G)| \geq 1+k+k(k-1) = k^2+1$, as desired. *Proof of the Claim:* For $y \in S$, y has no neighbors in S because they would form a 3-cycle with x, and the girth of G is 5. Thus, S_y does not overlap S. If S_y and $S_{y'}$ intersect in a vertex z, then x, y, z, y', x forms a 4-cycle — a contradiction, since the girth of G is 5. The claim is now proved.

The desired graphs are actually unique: C_5 for $k = 2$ and the Petersen graph for $k = 3$.

1.1.38 Let G be a simple graph in which every vertex has degree exactly 3.

First, suppose that G is bipartite, with partite sets A and B. Each vertex $x \in S$ is incident to exactly 3 edges, and these edges together with their endpoints form a claw; call it H_x. The claws H_x for $x \in A$ decompose G, which we show in two steps: (1) Each edge e of H_x is incident to x and a vertex in B, so e is not incident to any vertex $y \in A \setminus \{x\}$. Therefore $e \notin E(H_y)$. Since e is an arbitrary edge of H_x, $E(H_x) \cap E(H_y) = \emptyset$. (2) Any edge e of G has one endpoint in A. Since that vertex v is a center of a claw H_v that contains all edges incident to v, every edge e of G is in a claw H_v with $v \in A$.

For the other implication, suppose instead that G has a decomposition into claws $H_1, \ldots, H_k$. Let x_i be the center of H_i for each $1 \leq i \leq k$. Let $A = \{x_i \ : \ 1 \leq i \leq k\}$, and let $B = V(G) \setminus A$.

We show that A is an independent set: Suppose that $e = x_ix_j$ is an edge. Since $\deg_G(x_i) = 3$ and H_i is a claw, e must be in H_i. Similarly H_j contains e; so $E(H_i) \cap E(H_j) \neq \emptyset$ —a contradiction since $H_1, \ldots, H_n$ is a decomposition. Thus, A in an independent set.

We show that B is an independent set: By the decomposition, every edge e is in some claw H_i, incident to x_i. Thus no edge has both endpoints in B, so B is an independent set.