Solutions to Assignment #3, version 2.

1.3.12 If e is a cut-edge in graph G, then $G - e$ has a component H with exactly one endpoint of e. Since every vertex in G has even degree, $\sum_{u \in V(H)} \deg_G(u)$ is even, and $\sum_{u \in V(H)} \deg_H(u) = 2e(H)$ so it is even as well. But $\sum_{u \in V(H)} \deg_H(u) = \sum_{u \in V(H)} \deg_G(u) - 1$, because of the endpoint of e in H. Contradiction.

Alternatively:

Let G be an even graph with cut-edge e. The component that contains e is even and connected, so it has an Eulerian circuit T. But since T uses e exactly once, it cannot return to its start vertex, a contradiction.

Alternatively: An even graph decomposes into cycles, and an edge in a cycle is not a cut-edge.

Sharp example for $k \geq 1$:

Define G_k by adding edges to the larger partite set of $K_{2,2k}$, so that it becomes a $2k$-clique; then add edges from the small partite set to a vertex v. Let G'_k be the union of k copies of G_k, which all share the same vertex v (and are otherwise disjoint). Take two copies of G'_k and add an edge e between their two copies of v: this graph is $2k+1$-regular, and e is a cut-edge in it.

Another sharp example for $k \geq 1$:

Remove k edges from K_{2k+2} with $2k$ distinct endpoints, and add edges from those $2k$ vertices to a new vertex v. Take two copies of this graph and add an edge between their copies of v.

1.3.17 a. *Proof* Let x be a vertex of maximum degree in a graph G, and let $H = G - x$. Since H has exactly $d(x)$ fewer edges than G and $d(x) = \Delta(G)$, the average degree of H is $\frac{2[e(G)-\Delta(G)]}{n(G)-1}$. It is not greater than the average degree of G if and only if $\frac{2[e(G)-\Delta(G)]}{n(G)-1} \leq \frac{2e(G)}{n(G)}$. Using basic algebra, this inequality is equivalent to $e(G) \leq n(G)\Delta(G)$. This follows after verifying each of the following inequalities:

$$e(G) \leq 2e(G) = \sum_{v \in V(G)} d(v) \leq \sum_{v \in V(G)} \Delta(G) = n(G)\Delta(G)$$

b. *Disproof* A triangle is 2-regular, so its average degree and minimum degree are both 2. Every vertex has minimum degree. Deleting a vertex produces P_2, which has average degree 1.

1.3.26 Any two vertices in Q_3 differ in one, two or three positions. If we delete two vertices that differ in one position (adjacent vertices) or three positions ("opposite" vertices), we get a 6-cycle, but not otherwise. Thus there are $e(Q_3) = 12$ of the first type and $n(G)/2 = 4$ of the second type of cycle; 16 total.

A 3-dimensional subcube of Q_k is a copy of Q_3 induced by a set of vertices with fixed values on $k-3$ "coordinates", and which take all 2^3 possible values on the other three positions. Let C be a 6-cycle in Q_k.

Since C is a cycle, each coordinate changes an even number of times; if it changes at all, it changes at least twice. Therefore at most 3 coordinates change, and the other $k-3$ do not. When $k-2$ coordinates are fixed there are only 4 vertices, so that can't happen for C. Thus, exactly $k-3$ coordinates are fixed, which describes a unique 3-dimensional subcube of Q_k.

Alternatively: If C contains a path P of length 3 that changes three different coordinates, then the rest of C must change on those same three coordinates in order to return to the start vertex. In this case a unique 3-dimensional subcube contains C. Otherwise, every path of length 3 only changes on two coordinates. Letting $C = (v_1v_2v_3v_4v_5v_6)$, this means that the edges v_1v_2, v_3v_4 change on some position $i \in [k]$, and both v_2v_3, v_3, v_4 change on some position $j \in [k]$; but then $v_1 = v_4$, a contradiction.

(I removed the proof that was in the earlier version, which was not as good as these.)

The number of 3-dimensional subgraphs of Q_k is $\binom{k}{3}2^{k-3}$, so Q_k contains $\binom{k}{3}2^{k+1}$ six-cycles.

1.3.40 (a) If S is an independent set of size a in a simple graph G, then the $\binom{a}{2}$ pairs of vertices in G are not adjacent, but every other pair of vertices in G may be adjacent; the maximum number of edges occurs when these are all adjacent, so the maximum is $\binom{n}{2} - \binom{a}{2}$.

(b) Let $V_1, \ldots, V_k$ be the vertex sets of the components of a simple graph. Then $e(G) \leq \sum_{i \in [k]} \binom{V_i}{2}$, with equality when these sets are all cliques. If $V_i \geq V_j \geq 1$, then moving a vertex from V_j to V_i will increase $\sum_{i \in [k]} \binom{V_i}{2}$, because $[\binom{V_i+1}{2} + \binom{V_j-1}{2}] - [\binom{V_i}{2} + \binom{V_j}{2}] = |V_i| - |V_j| + 1 \geq 1$. Therefore the maximum is attained when all but one of the components is an isolated vertex, and the other component is a copy of $K_{n-(k-1)}$. Thus the maximum is $\binom{n-k+1}{2}$.

(c) G has at least two components, so the maximum is $\binom{n-k+1}{2}$ for some $k \geq 2$. This is largest for $k = 2$, so the maximum is $\binom{n-1}{2}$.

1.3.52 Following the proof of Theorem 1.3.23 on p41, observe that:

We have $\sum_{v \notin N(x)} d(v) \geq e(G)$, with equality when no edge has both endpoints in $V(G) - N(x)$. (Then $V(G) - N(x)$ is an independent set, so in this case G is bipartite with partite sets $N(x)$ and $V(G) - N(x)$.)

We have $\sum_{v \notin N(x)} d(v) \leq k(n-k)$, with equality when every vertex of $V(G) - N(x)$ has degree k.

We have $k(n-k) \leq \lfloor n^2/4 \rfloor$, with equality when k equals $\lceil n/2 \rceil$ or $\lfloor n/2 \rfloor$.

Since $e(G) \leq \sum_{v \notin N(x)} d(v) \leq k(n-k) \leq \lfloor n^2/4 \rfloor$, we have $e(G) = \lfloor n^2/4 \rfloor$ when all three inequalities are actually equal. If the first two are equal, then from the above observations, G is bipartite with partite sets of sizes k and $n-k$, and every vertex in the second set has degree k; since G is simple, it follows that $G \cong K_{k,n-k}$. If the third inequality is also equal, then we get $G \cong K_{\lceil n/2 \rceil, \lfloor n/2 \rfloor}$ or $G \cong K_{\lfloor n/2 \rfloor, \lceil n/2 \rceil}$ (which are isomorphic).