Solutions to Assignment #4.

1.4.3 Let W be a u, v-walk in a digraph G. If W has a vertex x that is repeated, then W contains an x, x-subwalk W', and replacing W' by x would change W' into a smaller u, v-walk. Therefore, a minimal u, v-walk in W has no repeated vertices, which means that it is a u, v-path.

1.4.4 Let W be a closed walk of odd length in a digraph G. If W has a repeated vertex x (other than first vertex = last vertex), then W contains an x, x-subwalk W'. If W' has odd length, apply induction to get an odd cycle with edges in W' (which are in $E(W')$). Otherwise, replace W' in W by x to create W''; then W'' has odd length so we can apply induction to W''. If W has no repeated vertex then W forms a cycle (this is the basis case).

Alternatively, the proof of Lemma 1.2.15 works verbatim.

1.4.5 If G contains a cycle C, then $G \setminus E(C)$ also has the property that $d^+(v) = d^-(v)$ for every vertex v (since both decrease by one if $v \in V(C)$ and both are unchanged if $v \notin V(C)$). Then apply induction to $G \setminus E(C)$ find a decomposition of it, and and add C to that list of graphs.

If $E(G) = \emptyset$ (basis case), then G is decomposed by an empty list. If $E(G) \neq \emptyset$, let P be a maximal path in G with last endpoint v. Since P has one edge incident to v, there must be an edge $vw \in E(G) \setminus E(P)$. Since P is a maximal path, $v \in V(P)$ and this edge completes a cycle in G. (Now proceed as above.)

Alternatively, use Lemma 1.4.23 or Theorem 1.4.24 as a tool to show that G contains a cycle.

1.4.13 (a) Suppose that two strong components H_1, H_2 intersect at a vertex x. Then for any vertices $u \in V(H_1)$, $v \in V(H_2)$, there is a u, x-path in H_1 and an x, v-path in H_2. Concatinating those paths gives a u, v-walk in $H_1 \cup H_2$, and that contains a u, v-path by exercise 1.4.3. Similarly, there is a v, u-path in $H_1 \cup H_2$. Also, if u, v are both in H_1, or both in H_2, then there is a u, v-path and a v, u-path since H_1 and H_2 are strong component. Therefore $H_1 \cup H_2$ is strongly connected. But then H_1 is not a maximal strongly connected subgraph of G, contradicting that H_1 is a strong component.

(b) Suppose that D^* has a cycle $v_0, e_1, \dots, e_k, v_k = v_0$ ($k \geq 2$ since D^* is loopless). Each edge e_i corresponds to an edge $x_i y_i \in D$ with endpoints in different strong components of D. Vertices y_{i-1}, x_i (modulo k) are contained in a strong component $D_{f(i)}$ for $i \in [k]$ (where f is is one-to-one function).

Let $H = \bigcup_{i \in [k]} D_{f(i)}$. Each $D_{f(i)}$ contains a y_{i-1}, x_i-path, and by alternating these paths with the edges $x_i y_i$, we obtain a closed walk W in H that contains x_i and y_i for all $i \in [k]$. We will show that H is strongly connected, contradicting the maximality of the strong components in H.

For any vertices u, v in H, there exists i, j such that $u \in V(D_{f(i)})$ and $v \in V(D_{f(j)})$. There is a u, y_{i-1}-path in $D_{f(i)}$ and an x_j, v-path in $D_{f(j)}$, Together with a y_{i-1}, x_i-walk within W, we get a u, v-walk in H. By exercise 1.4.3, this contains a u, v-path.

1.4.14 Let P be a maximal path in G and let v be its last vertex. Then v has no out-neighbors on P because G has no cycles, and v has no out-neighbors on $V(G) \setminus V(P)$ because P is maximal. Thus, $\deg^+(v) = 0$. Apply induction to $G - v$ to get an ordering $v_1, \dots, v_{n-1}$ of $V(G) - v$. Let $v_n = v$.

1.4.15 In any good path, the first coordinate must be increased exactly m times, and the second coordinate must be increased exactly n times. Therefore each good path has length $m+n$, and the exact choice of path depends on which of the $m+n$ edges advance the first coordinate, and which advance the second.

1.4.23 Let G be a graph. Repeatedly select a cycle remaining in G and delete its edges, until an acyclic graph G' remains. From G', repeatedly select a maximal path and delete its edges, until no edges remain. Let $\mathcal{S}$ be the set of cycles and paths that were chosen; this is a decomposition of G. Choose an orientation for each element of $\mathcal{S}$, and use that to orient each edge of G. Let D be the resulting digraph.

Each cycle contributes the same to the in-degree and out-degree of each vertex: both are one or both are zero. Each path likewise, except at its endpoints. At the moment when a path is deleted, its endpoints are leaves, and afterward they become isolated vertices. Therefore each vertex is an endpoint of at most one path in $\mathcal{S}$. Then $d_D^+(v) = d_D^-(v)$ if v is not the endpoint of a path in $\mathcal{S}$, and $|d_D^+(v) - d_D^-(v)| = 1$ otherwise.

1.4.25 (a) Let G be a connected graph. We'll prove a stronger statement by induction: For any vertex v of G, there is an orientation in which no vertex has odd out-degree, except perhaps v. The basis case is when $n(G) = 1$; orient each loop, and then the statement is true.

Let $H_1, \ldots, H_k$ be the components of $G - v$, and let v_i be a neighbor of v in H_i (for $i \in [k]$). Apply induction to orient each component H_i such that vertices of $V(H_i) - v_i$ have even degree. We can make v_i have even out-degree by correctly orienting the edge $v_i v$. Orient every other edge incident to v from v to its neighbor u; this does not change the out-degree of u. Now every edge of G is oriented and every vertex other than v has even out-degree.

Alternatively: Let G be a connected graph. Orient the edges of G arbitrarily. If u, v have odd out-degree, choose any u, v-path P in (the underlying graph) G and switch the orientation of each edge in P. The out-degree of each u and v changes by one, and the out-degree of each internal vertex of P changes by 2 or not at all. Thus, afterward exactly two more vertices of G have even out-degree. Repeat until there are fewer than two vertices with odd out-degree.

(b) *(Replaced previous proof)*

By part (a), at most one vertex has odd out-degree. $\sum d^+(u)$ is odd if there is exactly one vertex of odd out-degree, and $\sum d^+(u) = e(G)$ is even. Hence, every vertex has even degree. At each vertex v we can partition the edges with tails at v into pairs, since there are an even number of them. Every edge has its tail at a unique vertex, so this partitions $E(G)$ into pairs of incident edges, which form paths of length 2.