Math 454/553, Fall 2008, Instructor: Pelsmajer
Midterm exam, October 15: **solutions**

No books, no calculator, no talking, etc. Nothing but pens/pencils and blank paper. Please put all your answers on blank paper.

Read each question carefully. If you have any questions, ask me! Hints/facts are even possible, for a price (in points deducted).

1.a Finish the following definition: An edge in a graph is a cut-edge if and only if
removing it increases the number of components.

1.b Finish the following characterization: An edge in a graph is a cut-edge if and only if
it is not contained in a cycle.

2 State two (2) of the characterizations of trees (from class or the book).
State two of (A),(B),(C),(D) from Theorem 2.1.4 in West.

3 Prove that a graph G has at least $e(G) - n(G) + 1$ cycles.

Proof: Pick an edge in a cycle and delete it; repeat until there are no cycles. Say we deleted k edges; then G has at least k cycles. At the end we have a forest F, and (since F is can be extended to a spanning tree of G) $e(F) \leq n(G) - 1$. Then $e(G) \leq n(G) - 1 + k$, so $k \geq e(G) - n(G) + 1$.

4.a Finish the following statement: A connected digraph G is Eulerian if and only if
$d^+(v) = d^-(v)$ for all $v \in V(G)$.

4.b Prove the backward implication for problem **4.a**. (That is, assume that G is a connected digraph G with the special property you stated in problem **4.a**, and prove that G is Eulerian.)

Notes: (1) *Connected digraph* is not clear; I should have said *The underlying graph is connected.* You might have instead interpreted it to mean *The digraph is strongly connected*, which could make the problem a bit easier. I will accept either interpretation, as long as the proof is clear and correct. (2) Exercises 1.4.20 and 1.4.21 describe two ways to solve this problem.

Proof sketch: Suppose that G is a digraph and $d^+(v) = d^-(v)$ for all $v \in V(G)$, and the underlying graph is connected. Let T be a trail in G of maximum length. Let u and v be the first and last vertices of T, let E_T be the set of edges in T, let V_T be the set of vertices that appear (at least once) in T, and let G_T be the graph (V_T, E_T).

Each vertex has the same in-degree and out-degree in G_T, except for u and v when $u \neq v$. If $u \neq v$, then T has exactly one more edge to v than it has from v. Since $d^+_G(v) = d^-_G(v)$, there must be an edge in $E(G) - E_T$ from v. T could be extended along e, contradicting its maximum length. Therefore $u = v$, so T is a closed trail.

If the in-degree and out-degree of every vertex are the same in G_T as they are in G, then the underlying graph of G_T is a component of the underlying graph of G. Then G_T would equal G since the underlying graph of G is connected. Then $E_T = E(G)$, so T is an Eulerian circuit.

Thus we may assume that some vertex $v \in V_T$ is an endpoint of an edge e in $E(G) - E_T$. We may shift the start/end vertex of T to be v, obtaining a (new) closed trail with the same set of edges and vertices. If v is incident to e then we can extend the trail through e, and if e is incident to v then we can start a trail with e and then continue with T; either way, we obtain a longer trail in G, a contradiction.

5.a Let G be a simple graph not having P_4 or C_3 as an induced subgraph. Prove that G is a bipartite graph.

Proof: If G is not bipartite then G contains an odd cycle. Let C be an odd cycle of minimum odd length k. If $k = 3$ then $C \cong C_3$ and $G[V(C)] = C$, so G would contain an induced subgraph ismorphic to C_3, a contradiction. Then $k \geq 4$, and since k is odd, $k \geq 5$.

Two nonconsecutive vertices x, y along C cannot be endpoints of an edge, because then one of the x, y-paths along C together with the edge xy would form a smaller odd cycle. (Hence C is an induced cycle.) Then any four consecutive vertices along C induce a copy of P_4, a contradiction.

5.b Suppose that G is connected, bipartite, and simple, and it does not have P_4 or C_3 as an induced subgraph. Prove that G is a complete bipartite graph.

Proof: Let X, Y be partite sets of G. If G is not a complete bipartite graph, then there exists $x \in X$, $y \in Y$, such that $xy \notin E(G)$. Let P be a shortest x, y-path in G. Then any four consecutive vertices along P would induce a copy of P_4, so the length of P must be less than three. However the length must be odd since x and y are in opposite partite sets, and length one would mean that $xy \in E(G)$. So we have a contradiction. (We didn't need C_3-free as an extra fact since G is bipartite.)

6 Prove that a k-regular bipartite graph has no cut-edge, if $k \geq 2$. (Hint: I don't think the previous characterization is useful.)

Proof: Let H be a component of a k-regular bigraph which contains a cut-edge e. Let H_1, H_2 be the components of $H - e$. Let X_1, Y_1 be the partite sets of H_1; we may assume that the endpoint of e in H_1 is in Y_1. Then $e(H_1) = \sum_{v \in X_1} \deg_{H_1}(v) = k|X_1|$ and $e(H_1) = \sum_{v \in Y_1} \deg_{H_1}(v) = k|Y_1| - 1$, so $k(|Y_1| - |X_1|) = 1$. Since $k \geq 2$ and $|Y_1| - |X_1|$ is an integer, we have a contradiction.