The spectra of super line multigraphs

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Abstract

For an arbitrary simple graph G and a positive integer r, the super line multigraph of index r of G, denoted $\mathcal{M}_r(G)$, has for vertices all the r-subsets of edges. Two vertices S and T are joined by as many edges as pairs of distinct edges $s \in S$ and $t \in T$ share a common vertex in G. We present spectral properties of $\mathcal{M}_r(G)$ and particularly, if G is a regular graph, we calculate all the eigenvalues of $\mathcal{M}_r(G)$ and their multiplicities in terms of those of G.

Key words. super line graph, adjacency matrix, graph spectrum.
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1 Introduction

Bagga, Beineke, and Varma introduced the concept of super line graph in [3]. Given a simple graph (i.e. without loops or multiple edges) $G = (V, E)$, let $p = |V|$ be the number of vertices and $q = |E|$ be the number of edges. For a given positive integer $r \leq q$, the *super line graph of index* r of G is the graph $\mathcal{L}_r(G)$ which has for vertices all of the r-subsets of E, and two vertices S and T are adjacent whenever there exist distinct $s \in S$ and $t \in T$ such that s and t share a common vertex. Super line graphs are a generalization of line graphs. Indeed, from the definition it follows that $\mathcal{L}_1(G)$ coincides with the line graph $L(G)$. Properties of super line graphs were presented in [2, 4], and a good and concise summary can be found in [15]. A recent survey on several generalizations of line graphs can be found in [1]. More specifically, some results regarding the super line graph of index 2 were presented in [6] and [2].

This paper deals with super line multigraphs, a variation of the super line graph that was first defined and studied in [2]. Intuitively, the super line multigraph of a simple graph has the same set of vertices as the super line graph, but the number of edges between any two vertices S and T is given by the number of pairs $s \in S$ and $t \in T$ such that s and t are distinct and share a common vertex. Formally, for a given graph $G = (V, E)$ and a positive integer $r \leq q$, the *super line multigraph of index* r of G is the multigraph $\mathcal{M}_r(G)$ whose vertices are the r-subsets of $E(G)$, and two vertices S and T are joined by as many edges as ordered pairs of edges $(s, t) \in S \times T$ such that s and t are distinct and share a common vertex in G. Let $n = \binom{q}{r}$ be the number of vertices of $\mathcal{M}_r(G)$ and let m be the number of edges of $\mathcal{M}_r(G)$. In this paper, the underlying graph G is simple.

A simple graph G on p vertices labeled as $v_1, \ldots, v_p$ can be associated with an *adjacency matrix* $A = A(G)$, which is the $p \times p$ matrix whose entries $a_{i,j}$ are given by $a_{i,j} = 1$ if there is an edge joining v_i and v_j in G and $a_{i,j} = 0$ otherwise. If G has q edges labeled as $e_1, \ldots, e_q$, the *incidence matrix* $B = B(G)$ is the $p \times q$ matrix whose entries $b_{i,j}$ are given by $b_{i,j} = 1$ if e_j is incident to v_i, and 0 otherwise. Analogously, the entry $a_{i,j}$ of the adjacency matrix of a multigraph is the number of edges between v_i and v_j when $i \neq j$, and is the number of loops at v_i when $i = j$. The *characteristic polynomial* of the graph G, denoted by $\chi(G; \lambda)$, is defined as $\det(A - \lambda I)$. The *eigenvalues* of the graph G are those of A. The *algebraic multiplicity* of an eigenvalue α is the multiplicity of α as a root of the characteristic polynomial of A and is denoted by $m_a(\alpha, A)$ or $m_a(\alpha, G)$. The *geometric multiplicity* of an eigenvalue α is the dimension of $\ker(A - \alpha I)$ and is denoted by $m_g(\alpha, A)$ or $m_g(\alpha, G)$. If A is a real symmetric matrix, every eigenvalue α satisfies $m_a(\alpha, A) = m_g(\alpha, A)$. The *spectrum* of G is the set of eigenvalues of G together with their multiplicities as eigenvalues of A. The spectrum of a graph provides valuable information on its structure. More information on this topic can be found in [7, 9, 14].

The authors first investigated the spectra of super line multigraphs in [5]. In this paper we continue that study and give several new results as well extensions of some results in [5]. It is known that the eigenvalues of a line graph are greater than or equal to -2 (see [10] for properties of graphs with least eigenvalue -2). In this paper we generalize that result by giving a lower bound for the eigenvalues of any super line multigraph. Moreover, we characterize the graphs whose super line multigraph has the lower bound as an eigenvalue. This is done through the study of the relationships between the adjacency matrix and characteristic polynomial of a graph and the adjacency matrix and characteristic polynomial of its super line multigraph. Furthermore, those relationships allow us to give a complete description of the spectrum of the super line multigraph of a regular graph in terms of the spectrum of the original graph.

We refer the reader to [8, 12] for background on terminology or concepts not included here.

2 General results

Given a super line multigraph $\mathcal{M}_r(G)$ of index $r \leq q$ where $n = \binom{q}{r}$ represents its number of vertices, we denote by $U_{q,r}$ the $n \times q$ binary matrix whose rows are the binary strings of length q with exactly r entries equal to 1. Therefore, if G is a graph with q edges $e_1, \ldots, e_q$, the rows of the matrix $U_{q,r}$ represent all the possible r-subsets of edges, or the vertices of the super line multigraph $\mathcal{M}_r(G)$. We assume that the rows of $U_{q,r}$ are ordered "lexicographically" in the order of the edges $e_1, \ldots, e_q$. Thus, the first row (with ones in the first r entries) corresponds to the vertex $\{e_1, \ldots, e_r\}$ in $\mathcal{M}_r(G)$, while the last row (with ones in the last r entries) corresponds to the vertex $\{e_{q-r+1}, \ldots, e_q\}$. Whenever the parameters q and r are apparent from context, we will use U in place of $U_{q,r}$. Henceforth, for any matrix X, let $(X)_{ij}$ denote the entry in row i and column j of X, and let X^t denote the transpose of X. Before considering the spectrum of $\mathcal{M}_r(G)$, we first obtain a formula for the number of its edges.

Lemma 2.1 *Let $1 \leq r < q$. If G is a simple graph with q edges and p vertices whose degrees are given by the sequence $(d_1, \ldots, d_p)$, then the numbers of non-loop edges and loop edges, respectively, of $\mathcal{M}_r(G)$ are*

$$\left[\binom{q-1}{r-1}^2 - \binom{q-2}{r-2}\right] \sum_{i=1}^{p} \binom{d_i}{2}, \quad \textit{and} \quad 2\binom{q-2}{r-2} \sum_{i=1}^{p} \binom{d_i}{2}.$$

Proof: The proof is a straightforward extension of the proof for the case $r = 2$ of [2, Theorem 4.1]. Given a vertex v of degree d, consider a pair of edges $e, f \in \{e_1 \ldots, e_q\}$ in G that are incident at v. By renumbering if necessary, assume $e = e_1$ and $f = e_2$.

We count the number of non-loop edges of $\mathcal{M}_r(G)$ that arise due to the co-incidence of e_1, e_2 at v. These edges will have distinct end-points of the form $S = \{e_1, f_2, f_3, \ldots, f_r\}$ and $T = \{e_2, f_2', \ldots, f_r'\}$, When $\{e_1, e_2\} \nsubseteq S \cap T$, we may view S and T as being ordered, distinguished by $e_1 \in S$ and $e_2 \in T$. When $\{e_1, e_2\} \subseteq S \cap T$, the incidence of e_1 and e_2 gives rise to two edges of the form $\{S, T\}$: one from $e_1 \in S$ and $e_2 \in T$, and the other from $e_2 \in S$ and $e_1 \in T$. The total number of non-loop edges arising from the incidence of e_1 and e_2 is therefore obtained by selecting $\{f_2, \ldots, f_r\} \subseteq E(G) \setminus \{e_1\}$ and $\{f_2', \ldots, f_r'\} \subseteq E(G) \setminus \{e_2\}$, ignoring only the case $S = T$, and is therefore

$$\binom{q-1}{r-1}^2 - \binom{q-2}{r-2}.$$

This quantity multiplies by the $\binom{d}{2}$ pairs of edges $\{e, f\}$ incident at v.

The loop edges arising from the co-incidence of e_1 and e_2 are of the form $\{\{e_1, e_2, f_3, \ldots, f_r\}, \{e_1, e_2, f_3, \ldots, f_r\}\}$, where $f_3, \ldots, f_r$ are distinct edges in $E(G) \setminus \{e_1, e_2\}$. There are $\binom{q-2}{r-2}$ ways to select $f_3, \ldots, f_r$, and by definition, one instance of this form gives rise to two loops. The formula for loops follows since there are $\binom{d}{2}$ pairs of edges e, f incident at v. ■

Lemma 2.1 also holds for $r = q$, when $\mathcal{M}_r(G)$ is a 1-vertex graph with no non-loop edges. We will need the following lemma and its corollary. These were first presented without proof in [5]. For the sake of completeness we give the proofs here.

Lemma 2.2 *For a simple graph G, the adjacency matrix of the super line multigraph $\mathcal{M}_r(G)$ is given by*

$$A(\mathcal{M}_r(G)) = UA_LU^t$$

where A_L is the adjacency matrix of $L(G)$.

Proof: As above, let the edges of G be $e_1, \ldots, e_q$. Given two vertices i and j in $\mathcal{M}_r(G)$, we count the edges between i and j that arise due to an adjacency of e_k in j with edges of i, for $1 \leq k \leq q$. Now the $(i,j)^{th}$ entry of UA_LU^t is

$$(UA_LU^t)_{ij} = \sum_{k=1}^{q}(UA_L)_{ik}(U^t)_{kj} = \sum_{k=1}^{q}(UA_L)_{ik}(U)_{jk}.$$

Moreover, $(UA_L)_{ik} = \sum_{l=1}^{q}(U)_{il}(A_L)_{lk}$ counts the number of adjacencies of vertex i that arise due to the adjacency of an edge in this vertex with the edge e_k. On the other hand, $(U)_{jk} = 1$ iff the edge e_k is contained in vertex j. The result follows. ■

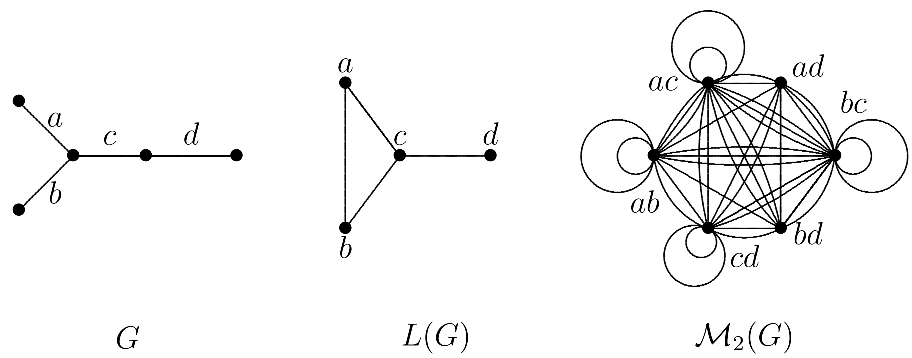

Figure 1: A graph G, its line graph $L(G)$ and super line graph $\mathcal{M}_2(G)$.

Note that for a simple graph G, $\mathcal{M}_r(G)$ may have loops, and consequently, $A(\mathcal{M}_r(G))$ may have nonzero entries in the diagonal. The next matrix computation illustrates how the Lemma 2.2 works in the example presented in Figure 1. The edge labels of G, in alphabetical order, are the indices of A_L.

$$A_L = \begin{pmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \qquad q = 4, r = 2;$$

$$U_{4,2} A_L U_{4,2}^t = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 \end{pmatrix};$$

$$A(\mathcal{M}_r(G)) = U_{4,2} A_L U_{4,2}^t = \begin{pmatrix} 2 & 3 & 1 & 3 & 1 & 2 \\ 3 & 2 & 2 & 3 & 3 & 2 \\ 1 & 2 & 0 & 3 & 1 & 2 \\ 3 & 3 & 3 & 2 & 2 & 2 \\ 1 & 3 & 1 & 2 & 0 & 2 \\ 2 & 2 & 2 & 2 & 2 & 2 \end{pmatrix}.$$

Corollary 2.3 *Under the same assumptions as in Lemma 2.2,*

$$A(\mathcal{M}_r(G)) = U_{q,r} B^t B U_{q,r}^t - 2 U_{q,r} U_{q,r}^t,$$

where B is the incidence matrix of G.

Proof: We observe that $B^t B = A_L + 2I$. The proof now follows from Lemma 2.2. ■

Observe that $U_{q,1}^t U_{q,1} = I$. Besides, if $2 \le r < q$, it was shown in [5] that the product $U_{q,r}^t U_{q,r}$ still has a particular pattern. We will give an expression for $U_{q,r}^t U_{q,r}$, for any r, $1 \le r < q$, that will be very useful in proving other results in this work. We need the following standard result from linear algebra, which we will apply to the eigensystems of UU^t and $U^t U$.

Lemma 2.4 *Let M and N be two (real or complex) matrices of sizes $m \times n$ and $n \times m$, respectively. Then* $\det(NM - xI_n) = (-x)^{n-m} \det(MN - xI_m)$.
■

Lemma 2.5 *For an integer $q \geq 2$ and an integer r, $1 \leq r < q$, let b and c be defined by*

$$\begin{array}{llll} b = \binom{q-2}{r-2} & \textit{and} & c = \binom{q-1}{r-1}, & \textit{if } r > 1 \\ b = 0 & \textit{and} & c = 1, & \textit{if } r = 1 \end{array}$$

Then

$$U_{q,r}{}^t U_{q,r} = bJ + (c-b)I,$$

where J is the all-ones matrix. Furthermore, both $U_{q,r}{}^t U_{q,r}$ and $U_{q,r} U_{q,r}{}^t$ have eigenvalue rc with multiplicity 1 corresponding to eigenvectors $\vec{1}_{n\times 1}$ and $\vec{1}_{q\times 1}$, respectively; and eigenvalue $c-b$ with multiplicity $q-1$; $U_{q,r}U_{q,r}^t$ also has eigenvalue 0 with multiplicity $n-q$.

Proof: U^tU is a $q \times q$ matrix whose entries are

$$(U^tU)_{ij} = \sum_{k=1}^{\binom{q}{r}} (U^t)_{ik}(U)_{kj} = \sum_{k=1}^{\binom{q}{r}} (U)_{ki}(U)_{kj}.$$

Therefore, for $i = j$, $(U)_{ki}$ and $(U)_{kj}$ coincide in exactly $\binom{q-1}{r-1}$ positions, and for $i \neq j$, those coincide in exactly $\binom{q-2}{r-2}$ positions. Thus the entries of the matrix U^tU are all $b = \binom{q-2}{r-2}$, except in the diagonal where all are $c = \binom{q-1}{r-1}$. For U^tU, $\vec{1}$ has eigenvalue rc, and all vectors orthogonal to $\vec{1}$ have eigenvalue $c-b$. Also for UU^t, $\vec{1}$ has eigenvalue rc, because for all $1 \leq i \leq q$,

$$UU^t\vec{1}(i) = \sum_{j=1}^{n}\sum_{k=1}^{q}(U)_{ik}(U)_{jk} = \sum_{k=1}^{q}(U)_{ik}\sum_{j=1}^{n}(U)_{jk}.$$

This last sum is rc, because for each of r 1's of row i of U, there are $\binom{q-1}{r-1}$ total rows j of U with a 1 in the same column. From Lemma 2.4, the nonzero portions of the spectra of U^tU and UU^t coincide. ■

Next we generalize the result [10, p. 5] that establishes that the eigenvalues of a line graph are greater than or equal to -2, and characterize the cases where the lower bound is attained in terms of a specific eigenvector.

Theorem 2.6 *Let G be a simple graph with q edges and incidence matrix $B = B(G)$, let r be an integer, where $1 \leq r < q$, and set $U = U_{q,r}$. If λ is an eigenvalue of $\mathcal{M}_r(G)$, then*

$$\lambda \geq -2\binom{q-2}{r-1}.$$

Furthermore, the lower bound is achieved iff there exists a nonzero vector $\phi \in \mathbb{R}^n$*, where* $n = \binom{q}{r}$*, such that*

(i) $\sum_{i=1}^n \phi(i) = 0$,

(ii) $BU^t\phi = \vec{0}$*, and*

(iii) $UU^t\phi = \binom{q-2}{r-1}\phi$.

Proof: The spectrum of UU^t is given by Lemma 2.5. Let $\{(\mu_i, \psi_i) : 1 \le i \le n\}$ be an orthonormal eigensystem for UU^t with eigenvalues $\mu_1 = r\binom{q-1}{r-1}$, $\mu_2 = \cdots = \mu_q = \binom{q-2}{r-1}$, and $\mu_{q+1} = \cdots = \mu_n = 0$, so that $\psi_1 = \vec{1}/\|\vec{1}\| = n^{-1/2}\vec{1}$. Note that $A(\mathcal{M}_r(G))$ is real symmetric and thus Hermitian. If λ is the smallest eigenvalue of $A(\mathcal{M}_r(G))$, then by the Rayleigh-Ritz Theorem ([13, p. 176]), we have $\lambda = \min_{\phi\in\mathbb{R}^n,\|\phi\|=1}\langle\phi, A(\mathcal{M}_r(G))\phi\rangle$. Since $\{\phi \in \mathbb{R}^n : \|\phi\| = 1\}$ is compact, let ϕ_0 be a vector achieving the minimum. Write $\phi_0 = \sum_{i=1}^n \alpha_i\psi_i$, so that

$$\begin{aligned}\lambda &= \left\langle \sum_{i=1}^n \alpha_i\psi_i, A(\mathcal{M}_r(G)) \sum_{i=1}^n \alpha_i\psi_i \right\rangle \\ &= \left\langle \alpha_1 n^{-1/2}\vec{1}, A(\mathcal{M}_r(G))\alpha_1 n^{-1/2}\vec{1} \right\rangle \\ &\quad + \left\langle \sum_{i=2}^n \alpha_i\psi_i, A(\mathcal{M}_r(G)) \sum_{i=2}^n \alpha_i\psi_i \right\rangle. \qquad (1)\end{aligned}$$

Since $\alpha_1^2 \ge 0$ and $A(\mathcal{M}_r(G))$ is a nonnegative matrix, the first term of (1) is zero, when $\phi_0 \perp \vec{1}$ or $A(\mathcal{M}_r(G)) = 0$, and otherwise positive. If $\sum_{i=2}^n \alpha_i\psi_i = \vec{0}$, the second term of (1) is zero and the lower bound follows. Otherwise, using Corollary 2.3 and the fact that $UU^t\psi_i = \mu_i\psi_i$, the second term is

$$\left\langle \sum_{i=2}^n \alpha_i\psi_i, UB^tBU^t \sum_{i=2}^n \alpha_i\psi_i \right\rangle - 2\left\langle \sum_{i=2}^n \alpha_i\psi_i, UU^t \sum_{i=2}^n \alpha_i\psi_i \right\rangle \qquad (2)$$

$$= \left\| BU^t \sum_{i=2}^n \alpha_i\psi_i \right\|^2 - 2\binom{q-2}{r-1}\sum_{i=2}^q \alpha_i^2 \qquad (3)$$

The lower bound follows since the first term of (2) is nonnegative, and the condition $\|\phi_0\| = 1$ implies that $\sum_{i=2}^q \alpha_i^2 \le 1$.

The lower bound is achieved if and only if $\|BU^t\sum_{i=2}^n \alpha_i\psi_i\|^2 = 0$ and $\sum_{i=2}^q \alpha_i^2 = 1$. Since $\sum_{i=2}^q \alpha_i^2 = 1$ iff $\alpha_1 = \alpha_{q+1} = \cdots \alpha_n = 0$, the lower bound is achieved iff $\phi_0 = \sum_{i=2}^q \alpha_i\psi_i$ and $BU^t\phi_0 = 0$, which is equivalent to conditions (i)-(iii). ■

By using the following result from [11] concerning the kernel of B we can characterize when $\mathcal{M}_r(G)$ achieves the lower bound in the theorem in terms of the structure of G.

Proposition 2.7 *Let G be a simple graph. There exists a nonzero vector $\theta \in \mathbb{R}^q$ such that $\sum_{i=1}^q \theta(i) = 0$ and $B\theta = \vec{0}$ iff G contains an even cycle or two edge-disjoint odd cycles in the same connected component.* ■

Corollary 2.8 *The super line multigraph $\mathcal{M}_r(G)$ of a simple graph G achieves the eigenvalue lower bound in Theorem 2.6 iff G contains an even cycle or two edge-disjoint odd cycles in the same connected component.*

Proof: Suppose that $\phi \in \mathbb{R}^n$ is a nonzero vector that achieves the lower bound and thus satisfies conditions (i)-(iii) of Theorem 2.6. Set $\theta = U^t\phi$. Then from (iii) we have $\theta \neq \vec{0}$; from (i), $\sum_{i=1}^q \theta(i) = \sum_{i=1}^q \sum_{j=1}^n (U^t)_{ij}\phi(j) = \sum_{j=1}^n \phi(j) \sum_{i=1}^q (U)_{ji} = \sum_{j=1}^n \phi(j)r = 0$; and, from (ii), $B\theta = \vec{0}$. Thus the forward direction follows from Proposition 2.7.

For the converse, by Proposition 2.7 suppose there exists a nonzero $\theta \in \mathbb{R}^q$ such that $\sum_{i=1}^q \theta(i) = 0$ and $B\theta = \vec{0}$. Using the same orthonormal eigensystem for UU^t as in the proof of Theorem 2.6, let $\mathcal{U} = \{U^t\psi_1, \ldots, U^t\psi_n\}$, so that $\mathcal{U}$ spans $\mathbb{R}^q$, as the eigenvalues $\mu_1, \ldots, \mu_q$ are nonzero by Lemma 2.5. $\mathcal{U}$ is an orthogonal basis for $\mathbb{R}^q$ since $\langle U^t\psi_i, U^t\psi_j\rangle = \langle UU^t\psi_i, \psi_j\rangle = \mu_i\langle\psi_i, \psi_j\rangle$ for $1 \leq i, j \leq q$. Now write $\theta = \sum_{i=1}^q \beta_i U^t\psi_i$ and note that $\beta_1 = 0$ because $\langle U^t\psi_1, \theta\rangle = n^{-1/2}\binom{q-1}{r-1}\langle\vec{1}, \theta\rangle = 0$. Set $\phi = \sum_{i=2}^q \beta_i\psi_i$ so that ϕ satisfies conditions (i)-(iii). ■

The following result, which originally appeared in [5], is a direct consequence of Lemma 2.4. We will need this in the next section.

Corollary 2.9 *For a simple graph G, and with notation as above, $\chi(UA_LU^t; \lambda) = (-\lambda)^{n-q}\chi(U^tUA_L; \lambda)$.* .■

Note that if one follows the definition of characteristic polynomial given in [7] the above proposition becomes $\chi(UAU^t; \lambda) = \lambda^{n-q}\chi(U^tUA_L; \lambda)$.

3 Regular graphs

In this section we consider a d-regular graph G with q edges, which has a $2(d-1)$-regular line graph $L(G)$. Let A_L be the adjacency matrix of $L(G)$, and let $\alpha_1 = 2(d-1), \alpha_2, \ldots, \alpha_q$ be the eigenvalues of A_L with corresponding eigenvectors $\phi_1 = \vec{1}, \phi_2, \ldots, \phi_q$, respectively. (Notice that $\vec{1}$ is an eigenvector because G, and therefore $L(G)$, are regular graphs.) Proposition 3.1 and Theorem 3.2 originally appeared in [5], but are included here for completeness, with simplified proof and determination of sign in Theorem 3.2.

Proposition 3.1 *Let G be a simple d-regular graph with q edges, and let A_L be the adjacency matrix of $L(G)$. Suppose that r is an integer, with $1 \leq r < q$, and define b and c as in Lemma 2.5. Let $\alpha_1, \ldots, \alpha_q$ be the eigenvalues of A_L corresponding to eigenvectors $\phi_1 = \vec{1}, \phi_2, \ldots, \phi_q$, respectively. Then the eigenvalues of U^tUA_L are $\lambda_1 = 2bq(d-1) + 2(c-b)(d-1)$ with eigenvector $\phi_1 = \vec{1}$; and, for $i = 2, \ldots, q$, $\lambda_i = (c-b)\alpha_i$ with eigenvector ϕ_i; and*

$$\chi(U^tUA_L; \lambda) = \prod_{i=1}^{q} (\lambda_i - \lambda).$$

Proof: From Lemma 2.5, $U^tU = bJ + (c-b)I$. Therefore, since $L(G)$ is $2(d-1)$-regular, $U^tUA_L = 2b(d-1)J + (c-b)A_L$. Furthermore, A_L and J have eigenvector $\vec{1}$ simultaneously, and so all other eigenvectors of A_L are also eigenvectors, and are in the null space, of J. ∎

Theorem 3.2 *Let G be a simple d-regular graph with q edges, r an integer, $1 \leq r < q$, and A_L the adjacency matrix of $L(G)$. Let $\alpha_1, \ldots, \alpha_q$ be the eigenvalues of A_L, where α_1 corresponds to the eigenvector $\vec{1}$. Let $n = \binom{q}{r}$. Then*

$$\chi(\mathcal{M}_r(G); \lambda) = (-\lambda)^{n-q} \prod_{i=1}^{q} (\lambda_i - \lambda),$$

where $\lambda_1 = 2bq(d-1) + 2(c-b)(d-1)$, and $\lambda_i = (c-b)\alpha_i$, for $i = 2, \ldots, q$. ∎

The previous theorem is a direct consequence of Corollary 2.9 and Proposition 3.1. It gives the spectrum of the super line multigraph $\mathcal{M}_r(G)$ for any r, $1 \leq r < q$, in terms of the spectrum of $L(G)$ in the case that G is d-regular.

Moreover, we shall give a formula for the spectrum of the super line multigraph $\mathcal{M}_r(G)$ in terms of the spectrum of G for some graphs. This is based on the following result by Sachs ([7, p. 19]).

Theorem 3.3 (Sachs) *Let G be a simple d-regular graph with p vertices and q edges and let $L(G)$ be the line graph of G. Then*

$$\chi(L(G); \lambda) = (-2 - \lambda)^{q-p} \chi(G; \lambda + 2 - d). \blacksquare$$

In other words, if $\beta \neq d$ is an eigenvalue of G, then $d - 2 + \beta$ is an eigenvalue of $L(G)$. This observation yields the following corollary of Theorem 3.2.

Corollary 3.4 *Let G be a simple d-regular graph with p vertices and q edges. Let the eigenvalues of the adjacency matrix of G be $d = \beta_1 \geq \cdots \geq \beta_p$, where β_1 corresponds to the eigenvector $\vec{1}$. Let r be an integer, $1 \leq r < q$, and let $n = \binom{q}{r}$. Then*

$$\chi(\mathcal{M}_r(G); \lambda) = (-\lambda)^{n-q} \prod_{i=1}^{q} (\lambda_i - \lambda),$$

where $\lambda_1 = 2bq(d-1) + 2(c-b)(d-1)$, $\lambda_i = (c-b)(d-2+\beta_i)$ for $i = 2, \ldots, p$, and $\lambda_{p+1} = \cdots = \lambda_q = -2(c-b)$. $\blacksquare$

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