# Two-batch liar games on a general bounded channel

Robert B. Ellis[1]     Kathryn L. Nyman[2]

[1]Illinois Institute of Technology

[2]Loyola University, Chicago

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# Outline

# Basic liar game setting

Two-person game:

1. Carole picks a number $x \in [n] := \{1, \ldots, n\}$
2. Paul asks $q$ questions to determine $x$:
   given $[n] = A_1 \dot\cup A_2 \dot\cup \cdots \dot\cup A_t$,
   for what $i$ is $x \in A_i$?

Playing optimally, Carole answers with an adversarial strategy; it's a perfect information game.

Catch: Carole is allowed to lie at most $k$ times.

# Example ternary game

$t = 3$ (Ternary coding).

- Paul partitions $[n] = A_1 \dot{\cup} A_2 \dot{\cup} A_3$ and asks "for what $i$ is $x \in A_i$?"
- Carole answers 1, 2, or 3

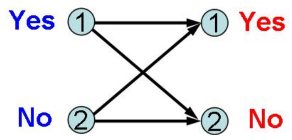
Example. $n = 6$, $q = 4$, $t = 3$, $k = 1$

| | Paul | | | | Lies | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| **Rnd** | $A_1$ | $A_2$ | $A_3$ | **Carole** | **1** | **2** | **3** | **4** | **5** | **6** |
| 1 | $\{1,2\}$ | $\{3,4\}$ | $\{5,6\}$ | 2 | ✓ | ✓ | | | ✓ | ✓ |
| 2 | $\{3\}$ | $\{4\}$ | $\{1,2,5,6\}$ | 3 | | | ✓ | ✓ | | |
| 3 | $\{1,2\}$ | $\{3,4\}$ | $\{5,6\}$ | 3 | ✓ | ✓ | ✓ | ✓ | | |
| 4 | $\{5\}$ | $\{6\}$ | $\emptyset$ | 1 | | | | | | ✓ |

Therefore $x = 5$.

# Binary symmetric case

- $t = 2$ binary case $\leftrightarrow$ "is $x \in A_1$?"
- symmetric lies: Carole may
  - lie with Yes when truth is No
  - lie with No when truth is Yes




**Question.** Given $q$, what is the maximum $n$ for which Paul has a winning strategy to find $x$?

- $k = 0$, binary search, $n = 2^q$

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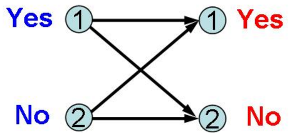


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- $k = 0$, binary search, $n = 2^q$
- $k = 1$, Pelc (1987)
- $k < \infty$, Spencer (1992) (*up to bounded additive error*)

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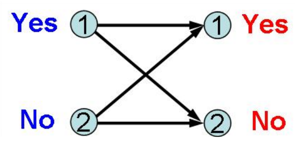


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- $k = 0$, binary search, $n = 2^q$
- $k = 1$, Pelc (1987)
- $k < \infty$, Spencer (1992) (*up to bounded additive error*)
- $k/q \to f \in (0, 1/2)$, Berlekamp (1962+); Alshwede, Deppe, Lebedev (2005) (*still partially open*)

# Binary symmetric case, $k = 1$

**Question.** Given $q$, what is the maximum $n$ for which Paul has a winning strategy to find $x$?

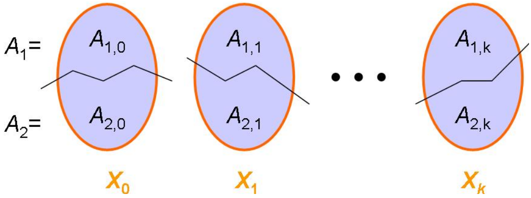
- Let $k = 1$, $y \in [n]$
- $q + 1$ ways for $y$ to be the distinguished element:

| | Game response string $w \in [2]^q$ | | | | | |
|---|---|---|---|---|---|---|
| 0 lies | $w_1$ | $w_2$ | $w_3$ | $\cdots$ | $w_{q-1}$ | $w_q$ |
| 1 lie | $\overline{w}_1$ | $*$ | $*$ | $\cdots$ | $*$ | $*$ |
| | $w_1$ | $\overline{w}_2$ | $*$ | $\cdots$ | $*$ | $*$ |
| | | $\vdots$ | | | $\vdots$ | |
| | $w_1$ | $w_2$ | $w_3$ | $\cdots$ | $w_{q-1}$ | $\overline{w}_q$ |

**Sphere Bound** $y, y'$ can't both be $x \implies n \leq 2^q / \binom{q}{\leq 1}$

# Binary symmetric case, $k < \infty$

$X_i :=$ elements of $[n]$ with $i$ accumulated lies




Paul balances $A_1 \dot{\cup} A_2$ by solving each round

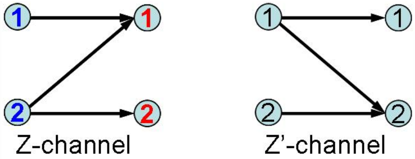
$$|A_1 \cap X_i| \doteq \frac{|X_i|}{2}, \qquad \text{for } 0 \le i \le k.$$

**Sphere Bound** $\binom{q}{\le k}$ ways for $y \in [n]$ to be the distinguished element
$\Longrightarrow n \le 2^q / \binom{q}{\le k}$

# Asymmetric lying

- asymmetric lies: Carole may
  - lie with Yes (1) when truth is No (2)
  - But not vice versa!

Called the $Z$-channel




- $k < \infty$, Dumitriu & Spencer (2004)
- $k < \infty$ w/improved asymptotics, Spencer & Yan (2003)

Asymmetric strategy: still based on balancing.

# A motivating question

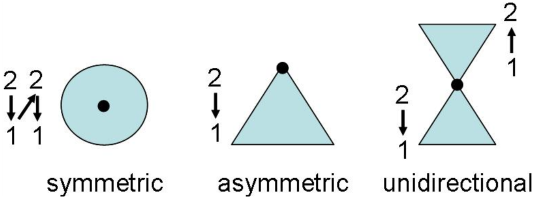
In 2005 Ioana Dumitriu was giving a talk on liar games,

and Nathan Linial asked:

What if Paul knows that Carole is lying according to one of the *Z*-channels, but not which one?

# A motivating question

Meanwhile, an equivalent question: What is the liar game version of packing/covering with unidirectional Hamming balls?




Our answer: Generalize the “channel” constraining Carole’s lies as much as possible.

# A closer look: game lie strings

| | **Paul** | | | **Carole** | **6's lie string** | |
|---|---|---|---|---|---|---|
| **Rnd** | $A_1$ | $A_2$ | $A_3$ | $w$ | $a$ | $b$ |
| 1 | $\{1,2\}$ | $\{3,4\}$ | $\{5,6\}$ | 2 | 3 | 2 |
| 2 | $\{3\}$ | $\{4\}$ | $\{1,2,5,6\}$ | 3 | | |
| 3 | $\{1,2\}$ | $\{3,4\}$ | $\{5,6\}$ | 3 | | |
| 4 | $\{5\}$ | $\{6\}$ | $\emptyset$ | 1 | 2 | 1 |

| | | | | | |
|---|---|---|---|---|---|
| Truthful string for $y = 6$ | $w' =$ | 3 | 3 | 3 | 2 |
| Lie string for $y = 6$ | $u =$ | 3<br>2 | | | 2<br>1 |
| Response string | $w =$ | 2 | 3 | 3 | 1 |

Write $u = (3,2)(2,1)$; we say $\boxed{w' \xrightarrow{u} w}$

# The general bounded $t$-ary channel

- Lies: $L(t) := \{(a,b) \in [t] \times [t] \ : \ a \neq b\}$ (truth= $a$, Carole: $b$)
- Lie strings: $L(t)^j := \{(a_1,b_1)\cdots(a_j,b_j) \ : \ (a_i,b_i) \in L(t)\}$
- Empty string: $L(t)^0 := \{\epsilon\}$

**Definition (General bounded channel)**

Fix $k \geq 0$. A channel $C$ of order $k$ is an arbitrary subset

$$C \subseteq \bigcup_{j=0}^{k} L(t)^j,$$

such that $C \cap L(t)^k \neq \emptyset$.

# Element survival and winning for Paul

**Definition**

An element $y \in [n]$ survives the game iff its lie string is in $C$.

**Definition**

Paul wins the original liar game iff at most one element survives after $q$ rounds.

Paul wins the pathological liar game iff at least one element survives after $q$ rounds.

$$\left.\begin{array}{r} A_C(q) := \max n \\ A^*_C(q) := \min n \end{array}\right\} \text{ such that Paul can win the } \left.\begin{array}{c} \text{original} \\ \text{pathological} \end{array}\right\} \text{ liar}$$

game with $n$ elements.

# Example channels

- Binary, symmetric, two lies. $(t = 2,\ k = 2)$

$$C = \{\epsilon,\ (1,2),\ (2,1),\ (1,2)(1,2),\ (1,2)(2,1),\ (2,1)(2,1),\ (2,1)(1,2)\}$$

$$\frac{2^q}{\binom{q}{\le 2}} - O(1) = A_C(q) \le A^*_C(q) = \frac{2^q}{\binom{q}{\le 2}} + O(1)$$

Guzicki ('90); E., Ponomarenko, Yan ('05)

- Binary, $Z$-channel, two lies. $(t = 2,\ k = 2)$

$$C = \{\epsilon,\ (2,1),\ (2,1)(2,1)\}$$

$A_C(q), A^*_C(q) \sim \dfrac{2^{q+\mathbf{2}}}{\binom{q}{\le 2}}$, Spencer, Yan ('03); here

# Example channels (con't)

- Binary, unidirectional, two lies. $(t = 2, k = 2)$

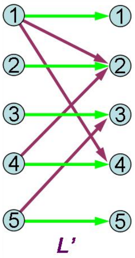
$$C = \{\epsilon, (1,2), (2,1), (1,2)(1,2), (2,1)(2,1)\}$$

$$A_C(q), A^*_C(q) \sim \frac{2^{q+1}}{\binom{q}{\leq 2}},$$ here

- Selective lies.
  - Pick arbitrary $L' \subseteq L(t)$.
  - Let $C = \bigcup_{j=0}^{k} (L')^j$.

$$A_C(q), A^*_C(q) \sim \frac{t^{q+k}}{|L'|^k \binom{q}{\leq k}}$$

Dumitriu, Spencer ('05); here



# The proposed sphere bound

- Select Paul's strategy tree to be entirely random partitions $[n] = A_1 \dot\cup \cdots \dot\cup A_t$
- The expected number of response strings for which $y$ survives is:

$$\sum_{u \in C} \binom{q}{|u|} t^{-|u|} \sim |C \cap L(t)^k| \binom{q}{k} t^{-k}.$$

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| Truthful string for $y$ | $w' =$ | $w'_1$ | $\cdots$ | $w'_{i_1}$ | $\cdots$ | $w'_{i_\ell}$ | $\cdots$ | $w'_{i_j}$ | $\cdots$ | $w'_q$ |
| Lie string for $y$ | $u =$ | | | $a_1$<br>$b_1$ | | $a_\ell$<br>$b_\ell$ | | $a_j$<br>$b_j$ | | |
| Response string | $w =$ | $w_1$ | $\cdots$ | $w_{i_1}$ | $\cdots$ | $w_{i_\ell}$ | $\cdots$ | $w_{i_j}$ | $\cdots$ | $w_q$ |

Compatibility: $\Pr(w'_{i_\ell} = a_\ell) = t^{-1}$

# The proposed sphere bound

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$$\sum_{u \in C} \binom{q}{|u|} t^{-|u|} \sim |C \cap L(t)^k| \binom{q}{k} t^{-k}.$$

**Definition (Asymptotic Sphere Bound)**

For $q$ rounds, base $t$, and an order $k$ channel $C$, the sphere bound is

$$\mathrm{SB}_C(q) := \frac{t^{q+k}}{|C \cap L(t)^k| \binom{q}{k}}.$$

# Carole's bound

## Theorem (Carole's bound)

$$\begin{aligned} A_C(q) &\leq \mathrm{SB}_C(q)(1+o(1)), \\ A^*_C(q) &\geq \mathrm{SB}_C(q)(1-o(1)). \end{aligned}$$

Proof idea.

- Most strings of $[t]^q$ are balanced.
- The response string set for which $y$ survives "looks random" when all its strings are balanced.
- $n$ too large $\Rightarrow$ response string sets collide
  too small $\Rightarrow$ response string sets fail to cover $[t]^q$

# Paul's bound

**Theorem (Paul's bound)**

$$\begin{aligned} A_C(q) &\geq \mathrm{SB}_C(q)(1-o(1)), \\ A_C^*(q) &\leq \mathrm{SB}_C(q)(1+o(1)). \end{aligned}$$

*Furthermore, **(1)** we may restrict Paul to two nonadaptive batches of questions of sizes $q_1$ and $q_2$, with*

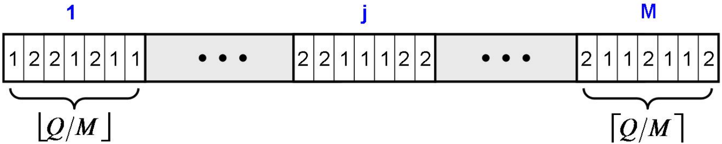
$$\begin{array}{rccl} & q_1+q_2 & =q & \text{and} \\ (\log_t q)^{3/2} << & q_2 & \leq \mathrm{cst}\cdot q^{k/(2k-1)}, & \end{array}$$

***(2)** the response sets for $A_C(q)$ are a subset of those for $A_C^*(q)$.*

Remark. Proof builds on techniques of Dumitriu&Spencer.

# $(M, r)$-balanced strings in $[t]^Q$

$$\frac{1}{t}\left\lceil \frac{Q}{M} \right\rceil - r(t-1) - \Theta(1) \le \#1's, \#2's \le \frac{1}{t}\left\lceil \frac{Q}{M} \right\rceil + r$$




## Lemma

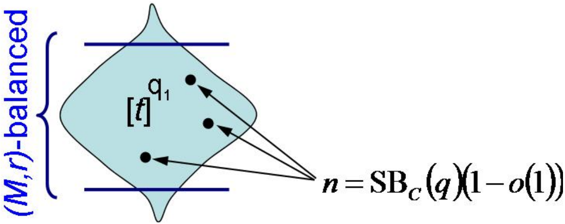
*Let* $u = (a_1, b_1) \cdots (a_j, b_j)$, *and* $w \in [t]^Q$ *be* $(M, r)$*-balanced. Then*

$$\binom{M}{j}\left(\frac{1}{t}\left\lceil \frac{Q}{M} \right\rceil - r(t-1) - \Theta(1)\right)^j \le |\{w' : w' \xrightarrow{u} w\}| \le \binom{M+j-1}{j}\left(\frac{1}{t}\left\lceil \frac{Q}{M} \right\rceil + r\right)^j,$$

$$\binom{Q}{j} t^{-j}(1 - o(1)) \le |\{w' : w' \xrightarrow{u} w\}| \le \binom{Q}{j} t^{-j}(1 + o(1)).$$

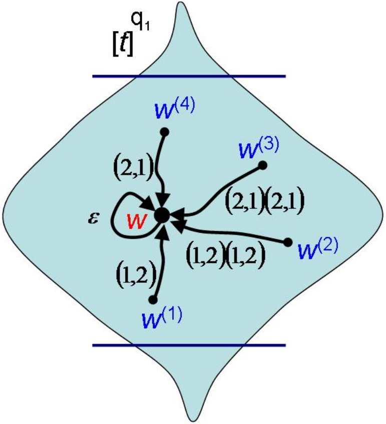
# First batch of $q_1$ questions

(Proof illustrated with $C = \{\epsilon, (1,2), (2,1), (1,2)(1,2), (2,1)(2,1)\}$.)




- Paul maps $n$ evenly to $(M, r)$-balanced vertices of $[t]^{q_1}$
- Paul partitions $[n]$ $q_1$ times based on each digit in mapping

# Carole's first batch response

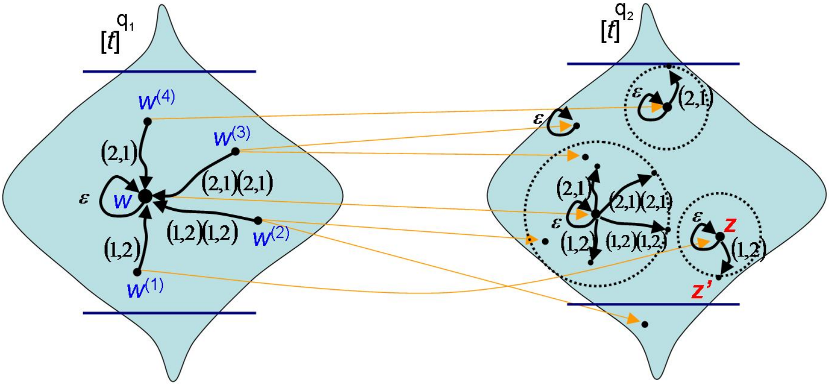



Suppose Carole responds with balanced $w \in [t]^{q_1}$.
Which $y \in [n]$ survive?

Any $y$ identified with $w'$ such that:

- $u \in C$, and
- $w' \xrightarrow{u} w$

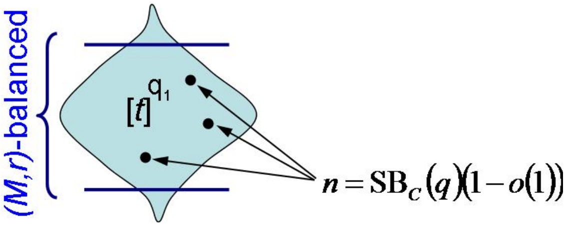
# Paul's second batch of $q_2$ questions




- $y$'s survive in various ways
- Fit $y$'s which can take more lies inside disjoint Hamming balls
- $(M, r)$-balance $\Rightarrow$ control on $|\{w^{(i)} : w^{(i)} \xrightarrow{u} w\}|$, $|\{z : z \xrightarrow{v} z'\}|$
- Greedily pack other $y$'s in unoccupied space

# First batch, pathological case

(Proof illustrated with $C = \{\epsilon, (1,2), (2,1), (1,2)(1,2), (2,1)(2,1)\}$.)



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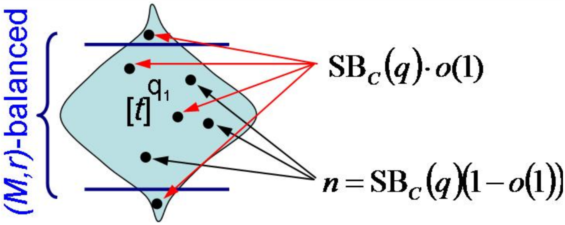
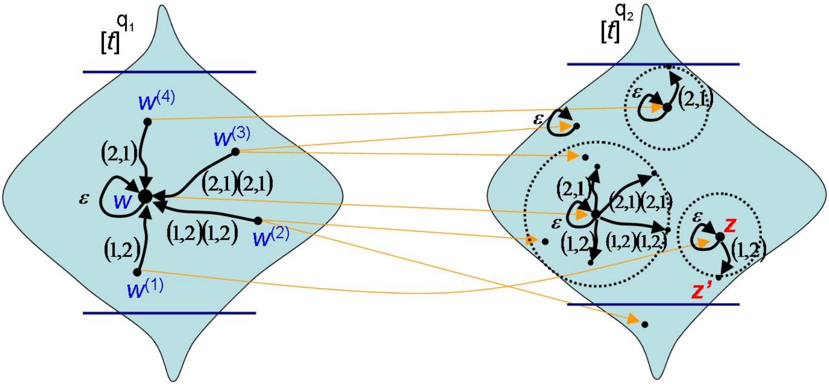



- Paul adds negligibly many elements evenly over $[t]^{q_1}$

# Paul's second batch, pathological case



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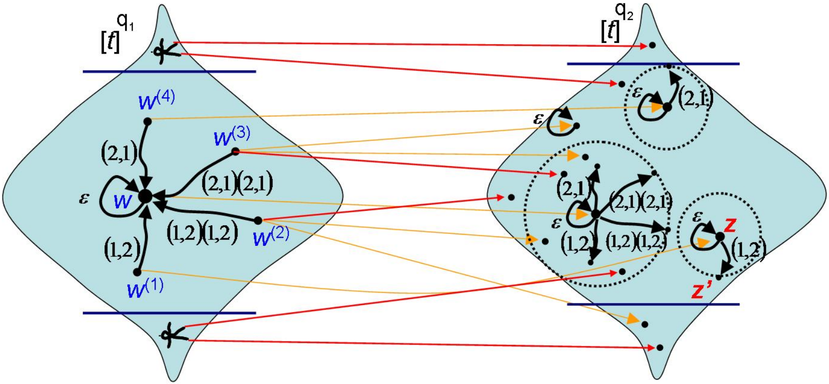


- Count only additional $y$'s for which Carole may not lie again
- Greedily convert packing into covering in $[t]^{q_2}$

# Summary

## Theorem

$$\mathrm{SB}_C(q)(1+o(1)) \geq A_C(q) \geq \mathrm{SB}_C(q)(1-o(1)),$$
$$\mathrm{SB}_C(q)(1-o(1)) \leq A_C^*(q) \leq \mathrm{SB}_C(q)(1+o(1)).$$

*Furthermore, **(1)** we may restrict Paul to two nonadaptive batches of questions of sizes $q_1$ and $q_2$, with*

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***(2)** the response sets for $A_C(q)$ are a subset of those for $A_C^*(q)$.*

# Concluding remarks and open questions

Open Questions.

- Can we further reduce or eliminate completely the adaptiveness?
- Can these techniques be used to improved the asymptotic best known packings and coverings of $[t]^q$ with fixed-radius Hamming balls (not tight for radius $\geq 2$)?
- Will these techniques work for coin-weighing, fault-testing, and related search problems?

Thank you very much.
Preprint at `http://math.iit.edu/~rellis/`.