Example 1: Solving a differential equation

In nuclear physics, the following fundamental relationship governs radioactive decay: the number of radioactive atoms N in a sample of a radioactive isotope decays at a rate proportional to N. This leads to the first order linear differential equation

$$\frac{dN}{dt} = -\lambda N$$

where λ is the decay constant. The Laplace transform can be used to solve this equation.

Rearranging the equation to one side, we have

$$\frac{dN}{dt} + \lambda N = 0.$$

Next, we take the Laplace transform of both sides of the equation:

$$\left(s\tilde{N}(s) - N_o\right) + \lambda\tilde{N}(s) \;=\; 0$$

where

$$\tilde{N}(s) = \mathcal{L}\{N(t)\}$$

and

$$N_o \;=\; N(0).$$

Solving, we find

$$\tilde{N}(s) = \frac{N_o}{s+\lambda}.$$

Finally, we take the inverse Laplace transform to find the general solution

$$\begin{aligned} N(t) &= \mathcal{L}^{-1}\{\tilde{N}(s)\} = \mathcal{L}^{-1}\left\{\frac{N_o}{s+\lambda}\right\} \\ &= N_o e^{-\lambda t}, \end{aligned}$$

which is indeed the correct form for radioactive decay.