Question. When are two groups the "same"?

Definition. An *isomorphism* ϕ from a group G to a group $\overline{G}$ is a bijection from G to $\overline{G}$ that preserves the operation. That is,

$$\phi(ab) = \phi(a)\phi(b) \qquad \text{for all } a, b \text{ in } G.$$

If there is an isomorphism from G onto $\overline{G}$, we say that G and $\overline{G}$ are *isomorphic* and write $G \equiv \overline{G}$.

(1) Suppose we were to write down the Cayley tables of two isomorphic groups G and $\overline{G}$. How can an isomorphism from G to $\overline{G}$ be described in terms of the Cayley tables?

For problems (2)-(4) exhaustively describe the isomorphisms for these small groups.

(2) Find an isomorphism from $\{-1, 1\}$ under multiplication to $\mathbb{Z}_2$.

(3) Find two distinct isomorphisms from the cyclic subgroup of rotations in D_3 to $\mathbb{Z}_3$.

(4) Let G be the group $\{(0,0), (1,0), (0,1), (1,1)\}$ under coordinate-wise addition mod 2. Find an isomorphism between G and the group generated by the 180 degree rotations of the tetrahedron. Is this isomorphism unique?

General Procedure for Proving Isomorphism

Step 1. Define the candidate mapping ϕ from G to $\overline{G}$.

Step 2. Prove that ϕ is one-to-one.

Step 3. Prove that ϕ is onto.

Step 2. Prove that for all $a, b \in G$, $\phi(ab) = \phi(a)\phi(b)$.

(continued on reverse)

Definition. An *automorphism* is an isomorphism from a group to itself.

(5) Prove that $\phi(x) = \sqrt{x}$ is an automorphism on $\mathbb{R}^+$, the group of positive real numbers under multiplication.

(6) Prove that $U(8)$ is not isomorphic to $U(10)$.

(7) Prove that S_4 is not isomorphic to D_{12}.