Group Members: ____

Definition: Coset of H in G
Let $a \in G$ and let H be a subgroup of a group G. The *left coset* of H in G containing a is

$$aH := \{ah \,|\, h \in H\},$$

and the *right coset* of H in G containing a is

$$Ha := \{ha \,|\, h \in H\}.$$

For later use, define $aHa^{-1} = \{aha^{-1} \,|\, h \in H\}$. If a coset contains a, then a is a *coset representative* of that coset.

(1) Refer to page 105 for this question. Let $G = A_4$ and let $H = \{\alpha_1, \alpha_5, \alpha_9\}$. Compute all of the left cosets of H in G in the following fashion:
(I) Pick an element a that has not appeared yet in any coset.
(II) Compute the coset containing this element a (multiply by every $h \in H$).
(III) Stop when all elements appear in a coset. Write your answers in the form $aH = \ldots$, $bH = \ldots$, etc.

(2) Repeat (1) except this time compute the right cosets of $\{\alpha_1, \alpha_5, \alpha_9\}$ in A_4.

(3) Determine the cosets of $2\mathbb{Z}$ (the even integers) in the integers $\mathbb{Z}$ (under addition). Does it matter if it is left or right cosets? Why or why not?

(4) Let G be a group with subgroup $H \leq G$. Prove that if $a \in Z(G)$, then $aH = Ha$.

Lemma: Properties of Cosets.
Let H be a subgroup of a group G, and let $a, b \in G$. Then,
1. $a \in aH$,
2. $aH = H$ iff $a \in H$,
3. $aH = bH$ or $aH \cap bH = \emptyset$,
4. $aH = bH$ iff $a^{-1}b \in H$,
5. $|aH| = |bH|$,
6. $aH = Ha$ iff $H = aHa^{-1}$,
7. $aH \leq G$ iff $a \in H$.

(5) How can we understand the Lemma part 2 from the permutations constructed in Cayley's Theorem? (Hint: consider the mapping $T_a : H \to H$ defined by $T_a(h) = ah$.)

(6) Every element $a \in G$ gives a left coset aH in G. Go back to (1) and compute the left cosets for those elements a that you didn't use already. Compare the results to the Lemma parts 3, 4, and 5.

(7) In questions (1-3), find the cosets for which $aH = Ha$.

Cosets partition G. From the Lemma we know that (i) the cosets of H in G are the same size, (ii) the union of the cosets of H in G is G, and (iii) cosets are *pairwise disjoint*. If G is finite there is a finite list of cosets. In this case the **set** of cosets $\{a_1H, a_2H, \ldots, a_rH\}$ **partition** G. If $a_1H, \ldots, a_rH$ are written without repeated cosets, then

$$\begin{aligned} \sum_{i=1}^{r} |a_iH| &= |G| \qquad \text{(all } a \in G \text{ appear in some coset)} \\ r \cdot |H| &= |G| \qquad \text{(all cosets have the same size),} \end{aligned}$$

and so the order of a subgroup H divides the order of the group G when $|G|$ is finite! This is Lagrange's Theorem.