Math 430-001 Group Activity 7B: Cosets and Lagrange's Theorem Fall 2008

Group Members: ____

Lagrange's Theorem: $|H|$ divides $|G|$
If G is a finite group and H is a subgroup of G, the $|H|$ divides $|G|$. Moreover, the number of distinct left (right) cosets of H in G is $|G|/|H|$.

Definition: Index of H in G
The *index* of H in G is written as $|G : H|$ and defined to be the number of cosets of H in G.

Corollary 1 of Lagrange's Theorem
If G is a finite group and H is a subgroup of G, then $|G : H| = |G|/|H|$.

(1) Prove that A_4 (with order 12) is a counterexample of the converse Lagrange's Theorem as follows. Assume there does exist an order 6 subgroup $H \leq A_4$. Take any order 3 element $a \in A_4$ and look at the cosets H, aH, and a^2H. What does the pigeonhole principle say about H, aH and a^2H? Deduce that in all possible cases that $a \in H$. How many order 3 elements are there? What is the contradiction?

(2) Prove Corollary 2 of Lagrange's theorem: in a finite group, the order of each element of the group divides the order of the group.

Corollary 3 of Lagrange's Theorem
A group of prime order is cyclic.

Corollary 4 of Lagrange's Theorem
Let G be a finite group, and let $a \in G$. Then $a^{|G|} = e$.

(4) Completely fill out the Cayley table for the group of order 6 which has an element a of order 3, and an element b of order 2 satisfying $ba = a^{-1}b$.

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Theorem 7.2: Classification of Groups of Order $2p$
Let G be a group of order $2p$, where p is a prime greater than 2. Then G is isomorphic to Z_{2p} or D_p.

Definition: Stabilizer of a Point
Let G be a group of permutations of a set S. For each $i \in S$, let $\text{stab}_G(i) = \{\phi \in G \mid \phi(i) = i\}$. We call $\text{stab}_G(i)$ the *stabilizer of i in G*.

Definition: Orbit of a Point
Let G be a group of permutations of a set S. For each $s \in S$, let $\text{orb}_G(s) = \{\phi(s) \mid \phi \in G\}$. The set $\text{orb}_G(s)$ is called the *orbit of s under G*.

(5) Let $G = A_4$ be the group of even permutations on $\{1, 2, 3, 4\}$. Compute $\text{stab}_G(1)$ and $\text{orb}_G(1)$. What is the product of the sizes of these two sets?

(6) Let G be D_4, the set of plane symmetries of the square with side length 2 centered at the origin. Let p be the point with Cartesian coordinates $(\sqrt{2}/2, \sqrt{2}/2)$. Compute $\text{stab}_G(p)$ and $\text{orb}_G(p)$. What is the product of the sizes of these two sets?

(7) Let G be a group of permutations of a set S and let $i \in S$. Prove that $\text{stab}_G(i)$ is a subgroup of G.