Group Members: ____

Theorem 8.3: $U(n)$ **as an External Direct Product**
Suppose s and t are relatively prime. Then

$$U(st) \approx U(s) \oplus U(t).$$

Moreover, $U_s(st) \approx U(t)$ and $U_t(st) \approx U(s)$.

Proof Sketch: The candidate isomorphism is $f : U(st) \to U(s) \oplus U(t)$ defined by

$$f(x) = (x \bmod s, x \bmod t). \tag{1}$$

For $U_s(st) \approx U(t)$, the candidate isomorphism is $g : U_s(st) \to U(t)$ defined by

$$g(x) = x \bmod t. \tag{2}$$

It remains to show that f and g are operation-preserving bijections.

(1) Write down the Cayley Table for $U(15)$.

(2) Theorem 8.3 states that $U(15) \approx U(3) \oplus U(5)$. Write out $f(x)$ in Equation (1) for all $x \in U(15)$.

(3) Theorem 8.3 states that $U_3(15) \approx U(5)$ and $U_5(15) \approx U(3)$. Just as in Question (2), write out $g(x)$ in Equation (2) for all x in both cases.

(4) Rewrite the Cayley table in Question (1) for $U(15)$, in the same order of columns and rows, except this time replacing the label $x \in U(15)$ with $f(x) = (x \bmod s, x \bmod t)$.

(5) Write down a group of the form $\mathbb{Z}_{n_1} \oplus \cdots \oplus \mathbb{Z}_{n_k}$ that is isomorphic to $U(15)$.

Theorem: Structure of $U(n)$
The groups $U(n)$ have the following structure. $U(2) = \{0\}$. $U(4) \approx \mathbb{Z}_2$.

$$\begin{aligned} U(2^n) &\approx \mathbb{Z}_2 \oplus \mathbb{Z}_{2^{n-2}}, && \text{for } n \geq 3; \\ U(p^n) &\approx \mathbb{Z}_{p^n - p^{n-1}}, && \text{for } p \text{ an odd prime.} \end{aligned}$$

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(6) Use the above theorems to write the following groups as the external direct products of cyclic groups: $U(10)$, $U(55)$, and $U(75)$.