Group Members: ______________________________________________

**Theorem: Index 2 Subgroups are Normal**
Let $|G|$ be a group and $H \leq G$ such that $|G : H| = 2$. Then $H \lhd G$.

**(1)** Prove the above theorem using the following steps.
**(a)** Write the form of the two left cosets of $H$ in $G$ as $H, aH$.
**(b)** Draw the simple diagram of the partition of $G$ into these left cosets. What must be true about $a$ to guarantee that $H \neq aH$?
**(c)** Now consider $H, Ha$. Use the fact you observed about $a$ to conclude that $H \neq Ha$. Why are these observations enough to conclude that $H \lhd G$? (Refer to p.138 if necessary.)

**(2)** For this question, refer to the above theorem, Theorem 9.2, and the Cayley table for $A_4$ on p.105. Assume $H \leq A_4$ and that $|H| = 6$ and arrive at a contradiction using the following steps. (Giving us a second proof that $A_4$ has no order 6 subgroup.)
**(a)** What does the above theorem say about $H$?
**(b)** Write the Cayley table for the factor group $A_4/H$ in terms of the left cosets of $A_4$.
**(c)** How many ways are there of writing the left coset $aH$, where $aH \neq H$?
**(d)** Compute $a^2H$ using the Cayley table and Theorem 9.2. Which coset is $a^2H$?
**(e)** Use p.138 to determine which coset $a^2$ lies in. Now look at the diagonal of the Cayley Table for $A_4$ to determine the size of the set $\{a^2 \mid a \in A_4\}$. Describe the contradiction that arises.

**Theorem 9.3: The $G/Z$ Theorem**
Let $G$ be a group and let $Z(G)$ be the center of $G$. If $G/Z(G)$ is cyclic, then $G$ is Abelian.

**Theorem 9.4:** $G/Z(G) \approx \text{Inn}(G)$
For any group $G$, $G/Z(G)$ is isomorphic to $\text{Inn}(G)$.

**Theorem 9.5: Cauchy's Theorem for Abelian Groups**
Let $G$ be a finite Abelian group and let $p$ be a prime that divides the order of $G$. Then $G$ has an element of order $p$.

**(3)** Prove that if $G$ is a non-Abelian group of order $pq$, where $p, q$ are primes, then $|Z(G)| = 1$. Use the following steps.
**(a)** What does Lagrange's Theorem say about the possibilities for $|Z(G)|$?
**(b)** Which possibility can you immediately rule out from the hypothesis on $G$?
**(c)** Which possibility do you not have to check due to the conclusion of the statement?
**(d)** For the other possibilities, what is $|G/Z(G)|$? Apply Corollary 3 on p.141 and Theorem 9.3.

**(4)** Use Theorem 9.4 to directly compute $\text{Inn}(D_4)$ by partitioning $D_4$ into the left cosets of $Z(D_4)$ and selecting one representative of each coset. See p.33.