Group Members: ____

Definition: Product of Subgroups
Let H, K be subgroups of a group G. Then the product HK is defined as

$$HK := \{hk \,|\, h \in H \text{ and } k \in K\}.$$

Definition: Internal Direct Product
Let H, K be subgroups of a group G. We say that G is the *internal direct product* of H and K, written $G = H \times K$, provided
(i) $H \lhd G$ and $K \lhd G$,
(ii) $G = HK$, and
(iii) $H \cap K = \{e\}$.

Facts when $G = H \times K$
- The order of G is $|H| \cdot |K|$.
- $G = H \times K \approx H \oplus K$.

Question. Why define internal direct products?
Answer. To decompose a group based on its internal structure.

Decomposition Technique. Look for two normal subgroups H, K of G.
Verify that $HK = G$.
Verify that $H \cap K = \{e\}$.
Note: only look for H and K with $|G| = |H| \cdot |K|$. By iterating inductively on H or K, we can get $H = H_1 \times H_2$ and $G = H_1 \times H_2 \times K$, etc., for the following definition and theorem.

Definition: $H_1 \times \cdots \times H_n$
Let $H_1, H_2, \ldots, H_n$ be a finite collection of normal subgroups of G. We say that G is the *internal direct product* of $H_1, H_2, \ldots, H_n$ and write $G = H_1 \times H_2 \times \cdots \times H_n$ if

1. $G = H_1 H_2 \cdots H_n = \{h_1 h_2 \cdots h_n \,|\, h_i \in H_i\}$, and
2. $(H_1 H_2 \cdots H_i) \cap H_{i+1} = \{e\}$ for $i = 1, 2, \ldots, n = 1$.

Theorem 9.6: $H_1 \times \cdots \times H_n \approx H_1 \oplus \cdots \oplus H_n$
If a group G is the internal direct product of a finite number of subgroups $H_1, \ldots, H_n$, then G is isomorphic to the external direct product of $H_1, \ldots, H_n$.

(1) For this question, $G = \mathbb{Z}_6$, $H = \langle 2 \rangle$, and $K = \langle 3 \rangle$.
(a) Determine whether $H, K \lhd G$.
(b) Write down all elements of HK.
(c) Write down the elements of $H \cap K$.
(d) Is $G = H \times K$?

(2) Same as (1) except $G = S_3$, $H = \langle(12)\rangle$, and $K = \langle(23)\rangle$.

(3) Same as (1) except $G = S_3$, $H = \langle(12)\rangle$, and $K = \langle(123)\rangle$.

(4) Let $G = \mathbb{Z}_{30}$ (note that $30 = 2 \cdot 3 \cdot 5$). Let $H_1 = \langle 15 \rangle$, $H_2 = \langle 10 \rangle$, and $H_3 = \langle 6 \rangle$.
(a) Determine whether $H_1, H_2, H_3 \lhd G$.
(b) Write down all elements of $H_1 H_2 H_3$. Use ellipses notation when the pattern is clear.
(c) Write down the elements of $H_1 \cap H_2$.
(d) Write down the elements of $H_1 H_2$. Use ellipses notation when the pattern is clear.
(e) Write down the elements of $(H_1 H_2) \cap H_3$.
(f) Is $G = H_1 \times H_2 \times H_3$ according to the definition?
(g) Why is this consistent with Corollary 2 of Theorem 8.2 on p.157?

Note: Question (4) introduces a major idea behind the classification of finite Abelian groups. $\mathbb{Z}_{30}$ has a known structure: we know there is an order 5 element $6 \in \mathbb{Z}_{30}$, and we can pull $\langle 6 \rangle$ out as an isomorphic copy of $\mathbb{Z}_5$.

Similarly, for an arbitrary finite Abelian group G, we can:
(i) Write $|G| = p^k m$, where $p \not| m$,
(ii) Find an element $x \in G$ with order p^j for j large as possible, and
(iii) Show that $G = \langle x \rangle \times K$ for a normal subgroup K of G with $|K| = mp^{k-j}$.
