Theorem 0.2. GCD Is a Linear Combination (integer linear combination).
For any nonzero integers a and b, there exist integers s and t such that

$$\gcd(a, b) = as + bt\,.$$

Moreover, $\gcd(a, b)$ is the smallest positive integer of the form $as + bt$.

Euclidean Algorithm. If a, b are positive integers, we may find $\gcd(a, b)$ by repeated use of the division algorithm to produce a decreasing sequence of integers $r_1 > r_2 > \cdots > r_k > 0$, where the last nonzero remainder r_k is equal to $\gcd(a, b)$.

$$\begin{aligned}
a &= bq_1 + r_1 & \qquad & 0 < r_1 < b \\
b &= r_1 q_2 + r_2 & & 0 < r_2 < r_1 \\
r_1 &= r_2 q_3 + r_3 & & 0 < r_3 < r_2 \\
&\ \vdots & & \ \vdots \\
r_{k-3} &= r_{k-2} q_{k-1} + r_{k-1} & & 0 < r_{k-1} < r_{k-2} \\
r_{k-2} &= r_{k-1} q_{k-1} + r_k & & 0 < r_k < r_{k-1} \\
r_{k-1} &= r_k q_{k+1} + 0 & &
\end{aligned}$$

Theorem 0.2. Euclid's Lemma.
If p is a prime that divides ab, then p divides a or p divides b.

Theorem 0.3. Fundamental Theorem of Arithmetic.
Every integer greater than 1 is a prime or a product of primes. This product is unique, except for the order in which the factors appear. Thus if

$$\begin{aligned}
n &= p_1 p_2 \cdots p_r\,, \qquad \text{and} \\
n &= q_1 q_2 \cdots q_s\,,
\end{aligned}$$

where the p's and q's are primes, then $r = s$ and, after renumbering the q's, we have $p_i = q_i$ for all i.

Proof of (9). Let S be the set of positive integers n satisfying

$$1+2+\cdots+n=\frac{n(n+1)}{2}.$$

<u>Base case.</u>
S contains 1 since $1=\frac{1(1+1)}{2}$.
<u>Inductive step.</u>
Let n be a positive integer and suppose that $n\in S$.
By this assumption,

$$1+2+\cdots+n=\frac{n(n+1)}{2}.$$

Adding $n+1$ to both sides, we have

$$\begin{aligned}
1+2+\cdots+n+(n+1) &= \frac{n(n+1)}{2}+(n+1)\\
1+2+\cdots+n+(n+1) &= \frac{n^2+n+2n+1}{2}\\
1+2+\cdots+n+(n+1) &= \frac{n^2+3n+1}{2}\\
1+2+\cdots+n+(n+1) &= \frac{(n+1)[(n+1)+1]}{2}.
\end{aligned}$$

Therefore $n+1\in S$.

By the First Principle of Mathematical Induction, S contains all positive integers. □

Proof of (10). By exhaustion, we analyze the small postage values to see which can be composed of 4 and 9 cent stamps.

1			15	
2			16	$4\cdot 4$
3			17	$1\cdot 9+2\cdot 4$
4	$1\cdot 4$		18	$2\cdot 9$
5			19	
6			20	$5\cdot 4$
7			21	$1\cdot 9+3\cdot 4$
8	$2\cdot 4$		22	$2\cdot 9+1\cdot 4$
9	$1\cdot 9$		23	
10			24	$6\cdot 4$
11			25	$1\cdot 9+4\cdot 4$
12	$3\cdot 4$		26	$2\cdot 9+2\cdot 4$
13	$1\cdot 9+1\cdot 4$		27	$3\cdot 9$
14			28	$7\cdot 4$

<u>Sketch of the rest of proof.</u> Argue that $24, 25, 26, 27\in S$, and for $n>27$, $n-4\in S\Rightarrow n\in S$. The Second Principle of Mathematical Induction gives the desired result, namely, that 23 is the largest amount that cannot be composed of 4 and 9 cent stamps.

Question. How is the question related to expressing

$$\begin{aligned}
1 &= 4\cdot s+9\cdot t,\\
-1 &= 4\cdot s'+9\cdot t' \quad ?
\end{aligned}$$