Theorem 6.2 Properties of Isomorphisms Acting on Elements.
Let G, $\overline{G}$ be groups with respective identities e, $\overline{e}$. Let $k, n \in \mathbb{Z}$ and $a, b \in G$. Then
1. $\phi(e) = \overline{e}$;
2. $\phi(a^n) = [\phi(a)]^n$;
3. $ab = ba$ iff $\phi(a)\phi(b) = \phi(b)\phi(a)$;
4. $G = \langle a \rangle$ iff $\overline{G} = \langle \phi(a) \rangle$;
5. $|a| = |\phi(a)|$; and
6. $|\{x \in G \,|\, x^k = b\}| = |\{x \in \overline{G} \,|\, x^k = \phi(b)\}|$.

Proof of 1.
We have

$$\begin{aligned} e &= ee && \text{by identity in } G \\ \phi(e) &= \phi(ee) = \phi(e)\phi(e) && \text{by operation preservation} \\ \overline{e}\phi(e) &= \phi(e)\phi(e) && \text{by identity in } \overline{G} \\ \overline{e} &= \phi(e) && \text{by right cancelation. } \square \end{aligned}$$

Proof of 2. (Induction)
For $n = 0$, $\phi(a^0) = \phi(e) = \overline{e} = [\phi(a)]^0$.
For $n = 1$, $\phi(a^1) = \phi(a) = [\phi(a)]^1$.
For $n = -1$, $\phi(a^{-1})\phi(a) = \phi(a^{-1}a) = \phi(e) = \overline{e}$, and therefore $\phi(a^{-1}) = [\phi(a)]^{-1}$.
Now assume $\phi(a^n) = [\phi(a)]^n$ for some positive integer n and consider $\phi(a^{n+1})$:

$$\begin{aligned} \phi(a^{n+1}) &= \phi(a^n a) = \phi(a^n)\phi(a) && \text{by operation preservation} \\ &= [\phi(a)]^n \phi(a) && \text{by inductive assumption} \\ &= [\phi(a)]^{n+1}. \end{aligned}$$

Therefore by induction $\phi(a^n) = [\phi(a)]^n$ for all positive integers n.
Now assume $\phi(a^n) = [\phi(a)]^n$ for some negative integer n and consider $\phi(a^{n-1})$:

$$\begin{aligned} \phi(a^{n-1}) &= \phi(a^n a^{-1}) = \phi(a^n)\phi(a^{-1}) && \text{by operation preservation} \\ &= [\phi(a)]^n [\phi(a)]^{-1} && \text{by inductive assumption and base case } n = -1 \\ &= [\phi(a)]^{n-1}. \end{aligned}$$

Therefore the statement holds for negative integer n, and thus for all $n \in \mathbb{Z}$.$\square$

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Proof of 3. We have that

$$\begin{aligned} ab = ba \quad &\text{if and only if} \quad \phi(ab) = \phi(ba) && \text{since } \phi \text{ is a bijective function} \\ &\text{if and only if} \quad \phi(a)\phi(b) = \phi(b)\phi(a) && \text{since } \phi \text{ is operation preserving.} \end{aligned}$$ □

Proof of 4. $(\Rightarrow)$
Assume that $G = \langle a \rangle$. We must show $\overline{G} = \langle \phi(a) \rangle$.
$(\supseteq)$ By definition $\phi(a) \in \overline{G}$, so by closure $\langle \phi(a) \rangle \subseteq \overline{G}$.
$(\subseteq)$ Let $b \in \overline{G}$.
Since ϕ is onto and $G = \langle a \rangle$, there exists some $k \in \mathbb{Z}$ with $\phi(a^k) = b$.
By **2**, $[\phi(a)]^k = b$, and so $b \in \langle \phi(a) \rangle$.
$(\Leftarrow)$ Assume that $\overline{G} = \langle \phi(a) \rangle$. We must show $G = \langle a \rangle$.
$(\supseteq)$ Since $a \in G$, by closure $\langle a \rangle \subseteq G$.
$(\subseteq)$ Let $b \in G$.
Since $\overline{G} = \langle \phi(a) \rangle$, there exists some $k \in \mathbb{Z}$ such that $\phi(b) = [\phi(a)]^k$.
By **2**, $\phi(b) = \phi(a^k)$. But ϕ is 1-1, so $b = a^k$.
Therefore $G = \langle a \rangle$. □

Proof of 5. The key is to note that when $\phi : G \to \overline{G}$ is restricted in domain to $\langle a \rangle$, it is an isomorphism from $\langle a \rangle$ to $\langle \phi(a) \rangle$. Then apply **4**. □

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Proof of 6. It suffices to show that $x \in G$ is a solution of $x^k = b$ in G iff $\phi(x) \in \overline{G}$ is a solution of $x^k = \phi(b)$ in $\overline{G}$. This is because ϕ is a bijection.

Let $x, b \in G$, and let $k \in \mathbb{Z}$. Then

$$\begin{aligned} x^k = b \quad & \text{if and only if} \quad \phi(x^k) = \phi(b) && \text{since } \phi \text{ is a bijective function} \\ & \text{if and only if} \quad [\phi(x)]^k = \phi(b) && \text{by } \mathbf{2}. \quad \square \end{aligned}$$