Group Members: ____

Two-line notation for permutations

A permutation $\alpha : \{1, \ldots, n\} \to \{1, \ldots, n\}$ is written in *two-line notation* as

$$\begin{bmatrix} 1 & 2 & \cdots & n-1 & n \\ \alpha(1) & \alpha(2) & \cdots & \alpha(n-1) & \alpha(n) \end{bmatrix} \tag{1}$$

The composition of permutations α and β in two line form is obtained in the function composition way by considering what happens to some element x in $\{1, \ldots, n\}$:

$$\begin{bmatrix} 1 & \cdots & n \\ \alpha(1) & \cdots & \alpha(n) \end{bmatrix} \begin{bmatrix} 1 & \cdots & n \\ \beta(1) & \cdots & \beta(n) \end{bmatrix} (x) \;=\; \alpha(\beta(x)). \tag{2}$$

Suppose $\alpha : \{1, 2, 3, 4, 5, 6\} \to \{1, 2, 3, 4, 5, 6\}$ is the permutation defined by $\alpha(1) = 2$, $\alpha(2) = 1$, $\alpha(3) = 3$, $\alpha(4) = 5$, $\alpha(5) = 4$, and $\alpha(6) = 6$; and $\beta : \{1, 2, 3, 4, 5, 6\} \to \{1, 2, 3, 4, 5, 6\}$ is the permutation defined by $\beta(1) = 6$, $\beta(2) = 1$, $\beta(3) = 2$, $\beta(4) = 4$, $\beta(5) = 3$, and $\beta(6) = 5$.

(1a) Write α and β in two-line form.

(1b) Write the composition $\alpha\beta$ in two line form by evaluating Eqn. 2 for all $x \in \{1, 2, 3, 4, 5, 6\}$ and writing the answer as a single permutation in the same form as Eqn. 1.

(1c) Do the same as in (1b), but this time for the permutation $\beta\alpha$ by composition in the opposite order.

(1d) Write the inverse permutation α^{-1} in the same two-line form as Eqn. 1 by parsing through the definition of α and the fact that $\alpha(x) = y \Leftrightarrow \alpha^{-1}(y) = x$.

One-line notation for permutations
One-line notation is the same as two-line notation except the top line is deleted. The permutation α on $\{1, \ldots, n\}$ is written in one-line form as:

$$\alpha \;=\; \left[\begin{array}{ccccc} \alpha(1) & \alpha(2) & \cdots & \alpha(n-1) & \alpha(n) \end{array}\right]. \tag{3}$$

(2) Let γ be the permutation on $\{1, 2, 3, 4, 5, 6, 7, 8\}$ defined by $\gamma(1) = 1$, $\gamma(2) = 3$, $\gamma(3) = 8$, $\gamma(8) = 4$, $\gamma(4) = 7$, $\gamma(7) = 2$, $\gamma(5) = 6$, and $\gamma(6) = 5$. Write γ in one-line notation.

(3) The inverse of a permutation given in one-line notation is often computed by (a) extending the notation to two-line notation, (b) swapping every column vertically, and (c) resorting the columns according to the top row. Carry out this process to compute γ^{-1} where γ is given in Eqn. 3.

(4) Let $\alpha = [4\,1\,3\,7\,5\,2\,6]$ and $\beta = [5\,1\,3\,6\,2\,7\,4]$. Compute $\alpha\beta$ and $\beta\alpha$ in the one-line notation form of Eqn. 3.

Cycle notation for permutations
Suppose we have a permutation α on $\{1, \ldots, n\}$ represented in the two-line form of Eqn. 1. To construct the cycle form of α, we do the following steps:
Step I. Pick any x not already in a cycle.
Step II. Write down the cycle $(x\ \alpha(x)\ \alpha^2(x)\ \cdots\ \alpha^{i(x)}(x))$, where $i(x)$ is the largest positive integer such that $x, \alpha(x), \ldots, \alpha^{i(x)}(x)$ are all distinct.
Step III. If not all numbers have been written down, go to Step I.
Step IV (optional). If there are cycles of length 1, they may be deleted if the domain of the permutation is known from context.

(5) Write down the cycle notation for α, β, $\alpha\beta$, $\beta\alpha$, and α^{-1} in **(1)**.

Break. Definition: For $n \in \mathbb{Z}^+$, we write S_n for the group of permutations of order n.

Theorem 5.1 Product of Disjoint Cycles. Every permutation of a finite set can be written as a cycle or as a product of disjoint cycles.