Real World Graph Efficiency

Bryan Ek*, Caitlin VerSchneider†, Joy Lind‡, and Darren A. Narayan*

Abstract

In a network it is often desirable to determine the shortest path between a given pair of nodes in the network, where the length of a path is equal to the number of edges comprising it. Longer paths are often inherently less "efficient" than shorter ones. In this paper we investigate the mathematical concept of efficiency of networks, and of star-like networks in particular. In this paper, we present exercises for students where they are introduced to the notion of efficiency. Students will also see an analysis of the Metropolitan Atlanta Rapid Transit Authority (MARTA) Subway system, that shows that this network is 82% as efficient as a network where there is a direct line between every pair of stations.

1 Introduction

In a network it is often desirable to determine the shortest path between a given pair of nodes in the network, where the length of a path is equal to the number of edges comprising it. Longer paths are often inherently less "efficient" than shorter ones. In this paper we investigate the mathematical concept of efficiency of networks, and of star-like networks, in particular. We apply these ideas to an analysis of the Metropolitan Atlanta Rapid Transit Authority (MARTA) Subway system, and show this network is 82% as efficient as a network where there is a direct line between every pair of stations.

In this paper, we define the distance $d(v_i, v_j)$ between any two vertices v_i and v_j in a graph to be the number of edges in a shortest path between

*School of Mathematical Sciences, Rochester Institute of Tech., Rochester, NY 14623
†Mathematics Department, Nazareth College, Rochester, NY 14618
‡Department of Mathematics, University of Sioux Falls, Sioux Falls, SD, 57105

v_i and v_j. If there is no path connecting v_i and v_j, then $d(v_i, v_j) = \infty$. In 2001, Latora and Marchiori introduced the measure of efficiency between vertices in a graph [1]. The (unweighted) *efficiency* between two vertices v_i and v_j is defined to be $\in_{i,j} = 1/d(v_i, v_j)$ for all $i \neq j$. We can obtain an overall measure of the graph by averaging the efficiency over all pairs of vertices. The *global efficiency* of a graph $E_{glob}(G) = \frac{1}{n(n-1)} \sum_{i \neq j} \in (v_i, v_j)$ is the average of the efficiencies over all pairs of the distinct n vertices.

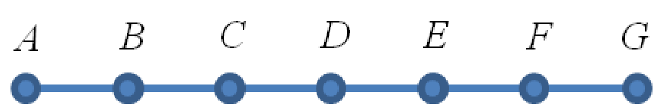
In 2002, Latora and Marchiori explored the global efficiency of the Boston Subway (MBTA) and found that the MBTA network is 63% as efficient as a network where there is a direct line between any two stations. Motivated by the design of the Metropolitan Atlanta Rapid Transportation Authority (MARTA) Subway network (see Figure 7), we investigate the global efficiency of subdivided stars. We show that networks of this type have a high level of efficiency. We apply these ideas to an analysis of the MARTA Subway system and show that its network is 82% as efficient as a network where there is a direct line connecting each pair of stations.

2 Efficiency

We begin with the definition of a path.

Definition 1 *Let P_n denote the path on vertices $v_1, v_2, ..., v_n$ with edges $v_1v_2, v_2v_3, ..., v_{n-1}v_n$. The distance $d(v_i, v_j)$ between distinct vertices v_i and v_j is $|i-j|$. Hence the efficiency between different vertices v_i and v_j is $\in (v_i, v_j) = \frac{1}{d(v_i,v_j)} = \frac{1}{|i-j|}$.*

Example 2 *Let $G = P_7$ with vertices A, B, C, D, E, F and G.*



The distances between pairs of vertices are given in the matrix below.

$DM =$

$L(G)$	A	B	C	D	E	F	G
A	0	1	2	3	4	5	6
B	1	0	1	2	3	4	5
C	2	1	0	1	2	3	4
D	3	2	1	0	1	2	3
E	4	3	2	1	0	1	2
F	5	4	3	2	1	0	1
G	6	5	4	3	2	1	0

The efficiency matrix is then as follows.

$EM =$

$E(G)$	A	B	C	D	E	F	G
A	0	1	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{4}$	$\frac{1}{5}$	$\frac{1}{6}$
B	1	0	1	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{4}$	$\frac{1}{5}$
C	$\frac{1}{2}$	1	0	1	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{4}$
D	$\frac{1}{3}$	$\frac{1}{2}$	1	0	1	$\frac{1}{2}$	$\frac{1}{3}$
E	$\frac{1}{4}$	$\frac{1}{3}$	$\frac{1}{2}$	1	0	1	$\frac{1}{2}$
F	$\frac{1}{5}$	$\frac{1}{4}$	$\frac{1}{3}$	$\frac{1}{2}$	1	0	1
G	$\frac{1}{6}$	$\frac{1}{5}$	$\frac{1}{4}$	$\frac{1}{3}$	$\frac{1}{2}$	1	0

We note that the matrix is symmetric about the main diagonal. To compute the global efficiency of this graph, we start by summing the elements in the upper triangle of the matrix: $6(1)+5(\frac{1}{2})+4\left(\frac{1}{3}\right)+3\left(\frac{1}{4}\right)+2\left(\frac{1}{5}\right)+1\left(\frac{1}{6}\right)$. Finally we divide by the number of non-diagonal elements. Therefore $E_{glob}(P_7) = \frac{1}{7\cdot 6}\cdot 2\left(\sum_{i=1}^{7-1}\frac{7-i}{i}\right) = \frac{223}{420}$.

The above example can be generalized as the next Theorem illustrates.

Theorem 3 $E_{glob}(P_n) = \frac{2}{n\cdot(n-1)}\left(\sum_{i=1}^{n-1}\frac{n-i}{i}\right)$

Proof. Consider the graph P_n with vertices $v_1, v_2, ..., v_n$. For all $1 \leq i \leq n-1$, there are $n-i$ pairs of vertices $\{(v_1, v_{i+1}), (v_1, v_{i+1}), ..., (v_{n-i}, v_n)\}$ whose distance between them is i. Summing over all i and dividing by the total number of pairs $\binom{n}{2}$ gives $E_{glob}(P_n) = \frac{2}{n\cdot(n-1)}\left(\sum_{i=1}^{n-1}\frac{n-i}{i}\right)$. ■

3 Efficiency under the Euclidean Metric

When analyzing the efficiency of a transportation network, it is natural to compare global efficiency under the graph metric $E_{glob}(G)$ versus a Euclidean metric $E^w_{glob}(G)$. The efficiency $E^w_{glob}(G)$ can be used to analyze real-world networks such as a subway system. When calculating $E^w_{glob}(G)$ the weight of an edge will be the Euclidean distance between the corresponding vertices. We can obtain a measure of how efficient the network is by dividing $E_{glob}(G)$ by $E^w_{glob}(G)$. This gives us the efficiency ratio $E_{Ratio}(G)$.

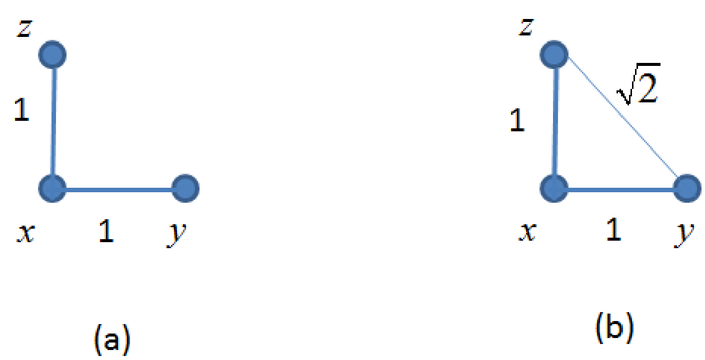


Figure 2. Comparison of unweighted efficiency (a) and weighted efficiency (b)

Exercise 4 *For the graph shown in Figure 2, determine $E_{glob}(G)$, $E^w_{glob}(G)$, and $E_{Ratio}(G)$.*

[Solution] For the unweighted efficiency we have $\in(x, y) = 1$, $\in(x, z) = 1$, and $\in(y, z) = \frac{1}{d(y,z)} = \frac{1}{2}$. Hence $E_{glob}(G) = \frac{1}{3\cdot 2} \cdot 2(1 + 1 + \frac{1}{2}) = \frac{5}{6}$ ≈ 0.83. However for the maximum weighted efficiency we have $\in(x, y) = 1$, $\in(x, z) = 1$, and $\in(y, z) = \frac{1}{\sqrt{2}}$. Hence $E^w_{glob}(G) = \frac{1}{3\cdot 2} \cdot 2(1 + 1 + \frac{1}{\sqrt{2}}) = \frac{1}{6}\sqrt{2} + \frac{2}{3} \approx 0.90$. By examining the ratio of the unweighted efficiency to the maximum weighted efficiency, we can compare how efficient a graph network is compared to a Euclidean network. The ratio $E_{Ratio}(G) = E_{glob}(G)/E^w_{glob}(G) = \frac{5}{6}/(\frac{1}{6}\sqrt{2} + \frac{2}{3}) \approx 0.92$. Hence for this particular graph, the first network is 92% as efficient as the second network. This means that the route from y to x to z is 92% as efficient as going directly from y to z.

Exercise 5 *Let P_n be a path on n vertices. Determine $E_{glob}(P_n)$, $E^w_{glob}(P_n)$, and $E_{Ratio}(P_n)$.*

[Solution] The case where $G = P_n$ is straightforward since the shortest distance between any points is a straight line. Hence $E_{glob}(P_n) = E^w_{glob}(P_n)$, and $E_{Ratio}(P_n) = 1$.

4 Efficiency of subdivided star graphs

In this section we consider the efficiency of star-like networks. These graphs provide a practical model for subway networks (see the MARTA network in Figure 7). The graph $K_{1,r}$ is called a star and is a complete bipartite graph with a single vertex in one part and r vertices in the other. We next recall the graph operation known as an edge subdivision.

Definition 6 *An edge subdivision is an operation that is applied to an edge uv where a new vertex w is inserted, and the edge uv is replaced by edges uw and wv. A subdivision H of a graph G is a graph that can be obtained by performing a sequence of edge subdivisions.*

We subdivide the edges of a star to produce a subdivided star, as defined below.

Definition 7 *Let $S_{d,l}$ be the subdivision of the star $K_{1,r}$ where each edge is replaced by a path with l vertices. The vertex of degree d is referred to as the center.*

The subdivided star $S_{4,3}$ is shown in Figure 3.

Exercise 8 *Determine $E_{glob}(S_{4,3})$, $E^w_{glob}(S_{4,3})$, and $E_{Ratio}(S_{4,3})$.*

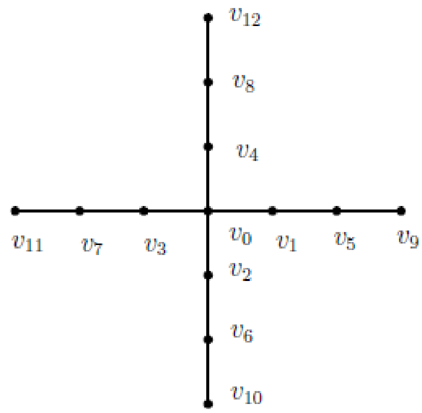


Figure 3. The subdivided star $S_{4,3}$

	v_0	v_1	v_2	v_3	v_4	v_5	v_6	v_7	v_8	v_9	v_{10}	v_{11}	v_{12}
v_0	0	1	1	1	1	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$
v_1		0	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	1	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$
v_2			0	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{3}$	1	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{4}$
v_3				0	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{3}$	1	$\frac{1}{3}$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{1}{4}$
v_4					0	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	1	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{2}$
v_5						0	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$	1	$\frac{1}{5}$	$\frac{1}{5}$	$\frac{1}{5}$
v_6							0	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{5}$	1	$\frac{1}{5}$	$\frac{1}{5}$
v_7								0	$\frac{1}{4}$	$\frac{1}{5}$	$\frac{1}{5}$	1	$\frac{1}{5}$
v_8									0	$\frac{1}{5}$	$\frac{1}{5}$	$\frac{1}{5}$	1
v_9										0	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$
v_{10}											0	$\frac{1}{6}$	$\frac{1}{6}$
v_{11}												0	$\frac{1}{6}$
v_{12}													0

Figure 4. Efficiency matrix for $S_{4,3}$

[Solution]. We first examine efficiencies between vertices on the same spoke including the center. Note that based on our labeling, there are three blocks of four identical entries across the top row and each continues in a 'downward diagonal pattern'. The total sum of these diagonals is: $\frac{4(3)}{1} + \frac{4(2)}{2} + \frac{4(1)}{3}$.

Next we examine efficiencies between vertices on different spokes. There are 'patches' of $\binom{4}{2} = 6$ identical entries. There is one patch where the entries are equal to $\frac{1}{2}$, two patches where the entries equal $\frac{1}{3}$, three patches where the entries equal $\frac{1}{4}$, two patches where the entries equal $\frac{1}{5}$, and one patch where the entries equal $\frac{1}{6}$. This pattern is inherent from the labeling of our vertices. The vertices $v_{h,i+1}, v_{h,i+2}, v_{h,i+3}$, and $v_{h,i+4}$, all have distance h from the center. We will consider paths between vertices on different spokes. Paths of length 2 must be between vertices where $h = 1$. Paths of length 3 must be between vertices where one vertex has $h = 1$ and another has $h = 2$. Paths of length 4 must be between vertices where both vertices have $h = 2$, or where one has $h = 1$ and the other has $h = 3$. Paths of length 5 must be between vertices where one vertex has $h = 2$ and another has $h = 3$. Paths of length 6 must be between vertices where $h = 3$. For each set of pairs of vertices there will be $\binom{4}{2}$ paths.

The sum over all of the patches is $\frac{4(3)}{2} \cdot \frac{1}{2} + \frac{4(3)}{2} \cdot \frac{2}{3} + \frac{4(3)}{2} \cdot \frac{3}{4} + \frac{4(3)}{2} \cdot \frac{2}{5} + \frac{4(3)}{2} \cdot \frac{1}{6}$. Using symmetry about the main diagonal, the total sum over all efficiencies is $2 \cdot \left(\frac{4(3)}{1} + \frac{4(2)}{2} + \frac{4(1)}{3} + \frac{4(3)}{2} \cdot \frac{1}{2} + \frac{4(3)}{2} \cdot \frac{2}{3} + \frac{4(3)}{2} \cdot \frac{3}{4} + \frac{4(3)}{2} \cdot \frac{2}{5} + \frac{4(3)}{2} \cdot \frac{1}{6} \right) = \frac{967}{15}$. Dividing by the number of non-diagonal entries in our matrix gives: $E_{glob}(S_{4,3}) = \frac{1}{13 \cdot 12} \cdot \frac{967}{15} = \frac{967}{2340} = 0.413\,25$.

When applying these methods in a real-world situation, we consider edges weighted by the Euclidean distance between the corresponding vertices (see Figure 5). For the non-weighted version we will consider the distance between any adjacent vertices to be 1. Furthermore, we consider all spokes to be linear and spaced at equal angles around the center vertex, v_0, in the plane.

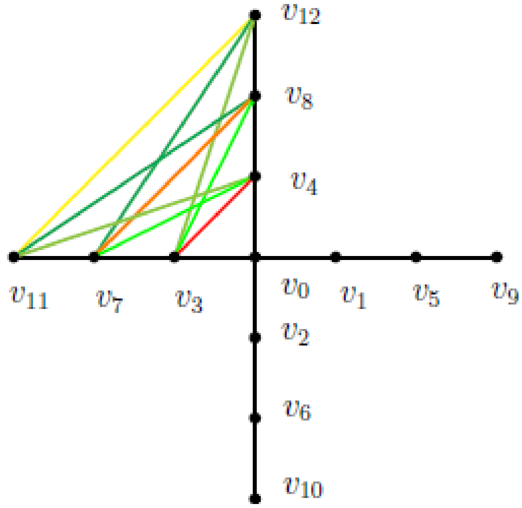


Figure 5. $S_{4,3}$ with some of the Euclidean distances drawn.

The following is a matrix of the efficiency of a subdivided star graph as if each vertex was connected with an edge weighted by the Euclidean distance between them. For example, v_8 and v_{11} would be connected by an edge of weight equal to the Euclidean distance between the points, $\sqrt{2^2+3^2} = \sqrt{13}$. Here the efficiency $\in (v_8, v_{11}) = \frac{1}{\sqrt{13}}$.

			$j=1$				$j=2$				$j=3$			
		v_0	v_1	v_2	v_3	v_4	v_5	v_6	v_7	v_8	v_9	v_{10}	v_{11}	v_{12}
	v_0	0	1	1	1	1	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$
i=1	v_1	1	0	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	1	$\frac{1}{\sqrt{5}}$	$\frac{1}{3}$	$\frac{1}{\sqrt{5}}$	$\frac{1}{2}$	$\frac{1}{\sqrt{10}}$	$\frac{1}{4}$	$\frac{1}{\sqrt{10}}$
	v_2	1	$\frac{1}{\sqrt{2}}$	0	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	$\frac{1}{\sqrt{5}}$	1	$\frac{1}{\sqrt{5}}$	$\frac{1}{3}$	$\frac{1}{\sqrt{10}}$	$\frac{1}{2}$	$\frac{1}{\sqrt{10}}$	$\frac{1}{4}$
	v_3	1	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	0	$\frac{1}{\sqrt{2}}$	$\frac{1}{3}$	$\frac{1}{\sqrt{5}}$	1	$\frac{1}{\sqrt{5}}$	$\frac{1}{4}$	$\frac{1}{\sqrt{10}}$	$\frac{1}{2}$	$\frac{1}{\sqrt{10}}$
	v_4	1	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	0	$\frac{1}{\sqrt{5}}$	$\frac{1}{3}$	$\frac{1}{\sqrt{5}}$	1	$\frac{1}{\sqrt{10}}$	$\frac{1}{4}$	$\frac{1}{\sqrt{10}}$	$\frac{1}{2}$
i=2	v_5	$\frac{1}{2}$	1	$\frac{1}{\sqrt{5}}$	$\frac{1}{3}$	$\frac{1}{\sqrt{5}}$	0	$\frac{1}{\sqrt{8}}$	$\frac{1}{4}$	$\frac{1}{\sqrt{8}}$	1	$\frac{1}{\sqrt{13}}$	$\frac{1}{5}$	$\frac{1}{\sqrt{13}}$
	v_6	$\frac{1}{2}$	$\frac{1}{\sqrt{5}}$	1	$\frac{1}{\sqrt{5}}$	$\frac{1}{3}$	$\frac{1}{\sqrt{8}}$	0	$\frac{1}{\sqrt{8}}$	$\frac{1}{4}$	$\frac{1}{\sqrt{13}}$	1	$\frac{1}{\sqrt{13}}$	$\frac{1}{5}$
	v_7	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{\sqrt{5}}$	1	$\frac{1}{\sqrt{5}}$	$\frac{1}{4}$	$\frac{1}{\sqrt{8}}$	0	$\frac{1}{\sqrt{8}}$	$\frac{1}{5}$	$\frac{1}{\sqrt{13}}$	1	$\frac{1}{\sqrt{13}}$
	v_8	$\frac{1}{2}$	$\frac{1}{\sqrt{5}}$	$\frac{1}{3}$	$\frac{1}{\sqrt{5}}$	1	$\frac{1}{\sqrt{8}}$	$\frac{1}{4}$	$\frac{1}{\sqrt{8}}$	0	$\frac{1}{\sqrt{13}}$	$\frac{1}{5}$	$\frac{1}{\sqrt{13}}$	1
i=3	v_9	$\frac{1}{3}$	$\frac{1}{2}$	$\frac{1}{\sqrt{10}}$	$\frac{1}{4}$	$\frac{1}{\sqrt{10}}$	1	$\frac{1}{\sqrt{13}}$	$\frac{1}{5}$	$\frac{1}{\sqrt{13}}$	0	$\frac{1}{\sqrt{18}}$	$\frac{1}{6}$	$\frac{1}{\sqrt{18}}$
	v_{10}	$\frac{1}{3}$	$\frac{1}{\sqrt{10}}$	$\frac{1}{2}$	$\frac{1}{\sqrt{10}}$	$\frac{1}{4}$	$\frac{1}{\sqrt{13}}$	1	$\frac{1}{\sqrt{13}}$	$\frac{1}{5}$	$\frac{1}{\sqrt{18}}$	0	$\frac{1}{\sqrt{18}}$	$\frac{1}{6}$
	v_{11}	$\frac{1}{3}$	$\frac{1}{4}$	$\frac{1}{\sqrt{10}}$	$\frac{1}{2}$	$\frac{1}{\sqrt{10}}$	$\frac{1}{5}$	$\frac{1}{\sqrt{13}}$	1	$\frac{1}{\sqrt{13}}$	$\frac{1}{6}$	$\frac{1}{\sqrt{18}}$	0	$\frac{1}{\sqrt{18}}$
	v_{12}	$\frac{1}{3}$	$\frac{1}{\sqrt{10}}$	$\frac{1}{4}$	$\frac{1}{\sqrt{10}}$	$\frac{1}{2}$	$\frac{1}{\sqrt{13}}$	$\frac{1}{5}$	$\frac{1}{\sqrt{13}}$	1	$\frac{1}{\sqrt{18}}$	$\frac{1}{6}$	$\frac{1}{\sqrt{18}}$	0

Figure 6. Euclidean efficiency matrix for $S_{4,3}$

Notice that the blocks of 4 identical terms with diagonals directed downward are identical to those appearing in the non-weighted case. These are the efficiencies between vertices on the same spoke or the center. For the pairs of vertices on different spokes, we focus on the squares which represent efficiencies between two vertices, where one is distance i from the center.

To obtain $E^w_{glob}(S_{4,3})$ we simply sum the entries of the matrix in Figure 6.

Hence $E^w_{glob}(S_{4,3}) = \frac{2}{13\cdot 12} * 12(1) + 8(\frac{1}{2}) + 4(\frac{1}{3}) + 2\left(\frac{1}{2}\right) + 4\left(\frac{1}{3}\right) + 4\left(\frac{1}{4}\right) + 4\left(\frac{1}{5}\right) + 2\left(\frac{1}{6}\right)$

$+4\left(\frac{1}{\sqrt{2}}\right) + 4\left(\frac{1}{\sqrt{8}}\right) + 4\left(\frac{1}{\sqrt{18}}\right) + 8\left(\frac{1}{\sqrt{5}}\right) + 8\left(\frac{1}{\sqrt{13}}\right) + 8\left(\frac{1}{\sqrt{10}}\right) = \approx 0.452\,72.$

We can generalize this example to the family of graphs $S_{d,l}$, which has d spokes each of length l.

Exercise 9 *What happens to the efficiency ratio of $S_{d,l}$ when l is fixed and d increases?*

Solution 10 *When d increases, the efficiency ratio decreases, as there are more vertices that are not on the same spoke. Efficiency decreases as the number of pairs of vertices on different spokes is increased.*

Exercise 11 *What happens to the efficiency ratio of $S_{d,l}$ when d is fixed and l increases?*

Solution 12 *As l increases, the efficiency ratio increases. To see why this is true note that a straight line path has an efficiency ratio of* 1*. We note that as the lengths of the spokes increases, the overall shape of a subdivided star becomes skewed and bears a closer resemblance to a path.*

5 Metropolitan Atlanta Rapid Transit Authority Subway

The Metropolitan Atlanta Rapid Transit Authority Subway has 38 stations shown in the map below.

Figure 7. Metropolitan Atlanta Rapid Transit Authority Map (www.itsmarta.com)

After obtaining rail distances along each of the lines directly from MARTA, we calculated the rail distance between every pair of stations. These distances (in miles) are given in the table below.

	NS	SS	D	MC	BH	LC	LNX	BHOG	CH	DRA	Arts	MT	NA	CVC	PCH	SPT	GS	K	I	EW	EL	DCT	AVD	KNS	IND	GNT	WE	OAK	LW	EP	COL	APT	DOME	VC	ASH	WL	HAM	BNK	
NS	0	1.04	1.86	2.31	6.76	8.42	6.93	5.98	4.97	5.31	10.9	11.44	12.07	12.45	12.91	13.34	13.58	13.55	12.98	12.69	12.7	12.3	12.52	13.39	14.21	13.72	14.79	16.19	17.38	19.14	20.97	21.68	13.24	13.3	13.24	14.21	14.61	12.64	NS
SS		0	0.83	1.3	5.84	7.5	5.97	4.92	3.94	4.49	10.02	10.59	11.21	11.57	12.04	12.45	12.66	12.62	12	11.71	11.67	11.25	11.49	12.36	13.19	12.84	13.94	15.25	16.54	18.31	20.13	20.84	12.36	12.43	12.66	13.46	13.92	11.86	SS
D			0	0.68	5.18	6.84	5.26	4.19	3.2	3.95	9.39	9.95	10.57	10.94	11.37	11.79	12	11.93	11.27	10.94	10.88	10.44	10.66	11.54	12.37	12.19	13.33	14.75	15.93	17.7	19.52	20.34	11.73	11.82	12.08	12.94	13.46	11.33	D
MC				0	4.55	6.2	4.67	3.67	3.15	4.16	8.73	9.3	9.91	10.28	10.73	11.15	11.36	11.31	10.69	10.39	10.39	10.01	10.28	11.21	12.11	11.55	12.66	14.09	15.27	17.04	18.86	19.57	11.08	11.16	11.41	12.26	12.77	10.65	MC
BH					0	1.67	0.58	1.86	4.48	6.26	4.2	4.76	5.37	5.75	6.2	6.63	6.82	6.79	6.3	6.13	6.52	6.49	7.05	8.44	9.63	7.01	8.15	9.59	10.76	12.53	14.35	15.05	6.55	6.63	6.94	7.96	8.64	6.31	BH
LC						0	1.63	3.09	5.71	7.45	2.57	3.12	3.73	4.1	4.55	4.97	5.16	5.12	4.67	4.57	5.18	5.37	6.09	7.61	8.88	5.36	6.55	8	9.16	10.94	12.74	13.43	4.92	5.05	5.41	6.58	7.41	4.93	LC
LNX							0	1.51	4.18	5.94	4.2	4.75	5.35	5.71	6.16	6.57	6.75	6.67	6.06	5.83	6.1	6	6.53	7.89	9.06	6.98	8.17	9.62	10.78	12.56	14.37	15.05	6.54	6.66	7.03	8.14	8.89	6.48	LNX
BHOG								0	2.67	4.44	7.52	6.16	6.73	7.34	7.58	7.95	8.08	7.95	7.16	6.8	6.75	6.41	6.75	7.87	8.91	8.36	9.6	11.05	12.2	13.47	15.76	16.44	7.95	8.09	8.5	9.66	10.42	8	BHOG
CH									0	1.78	8.24	8.72	9.3	9.62	10.04	10.46	10.55	10.36	9.4	8.9	8.47	7.84	7.89	8.55	9.3	10.88	12.15	13.61	14.73	16.5	18.27	18.92	10.5	10.67	11.11	12.32	13.09	10.66	CH
DRA										0	9.95	10.41	10.88	11.21	11.59	12.08	12.14	11.91	10.86	10.31	9.68	8.91	8.78	9.17	9.68	12.5	13.79	15.25	16.36	18.11	19.86	20.49	12.14	12.34	12.81	14.06	14.84	12.4	DRA
ARTS											0	0.59	1.24	1.61	2.09	2.45	2.71	2.81	2.97	3.3	4.6	5.3	6.19	7.85	9.19	2.82	3.97	5.42	6.58	8.36	10.17	10.86	2.35	2.46	2.87	4.23	5.23	2.67	ARTS
MT												0	0.62	1.01	1.47	1.87	2.12	2.24	2.54	2.98	4.39	5.19	6.1	7.76	9.1	2.25	3.44	4.9	6.05	7.83	9.63	10.31	1.8	1.95	2.45	3.93	5.01	2.49	MT
NA													0	0.32	0.84	1.25	1.5	1.66	2.23	2.81	4.32	5.22	6.13	7.78	9.11	1.63	2.87	4.33	5.46	7.24	9.03	9.71	1.22	1.43	2.04	3.64	4.78	2.37	NA
CVC														0	0.52	0.93	1.13	1.33	2.1	2.75	4.31	5.25	6.18	7.8	9.11	1.28	2.56	4.01	5.13	6.9	8.69	9.36	0.9	1.18	1.86	3.52	4.7	2.4	CVC
PCH															0	0.45	0.67	0.96	2.01	2.73	4.32	5.32	6.25	7.83	9.14	0.86	2.19	3.63	4.72	6.49	8.26	8.93	0.63	0.99	1.74	3.43	4.64	2.51	PCH
SPT																0	0.45	0.96	2.25	3.01	4.6	5.63	6.55	8.11	9.4	0.41	1.75	3.18	4.26	6.03	7.81	8.48	0.39	0.76	1.5	3.17	4.39	2.46	SPT
GS																	0	0.53	1.94	2.72	4.31	5.36	6.27	7	9.08	0.59	1.86	3.21	4.32	5.96	7.72	8.35	0.84	1.18	1.4	3.52	4.76	2.91	GS
K																		0	1.44	2.23	3.79	4.86	5.76	7.29	8.56	1.14	2.34	3.61	4.56	6.26	7.97	8.58	1.34	1.7	2.43	4.07	5.3	3.4	K
I																			0	0.78	2.35	3.42	4.32	5.87	7.15	2.56	3.79	5.02	5.91	7.53	9.17	9.73	2.6	2.98	3.74	5.42	6.64	4.49	I
EW																				0	1.59	2.63	3.54	5.11	6.4	3.32	4.57	5.79	6.65	8.24	9.85	10.4	3.34	3.71	4.48	6.16	7.38	5.15	EW
EL																					0	2.32	3.22	4.77	6.07	4.9	6.11	7.25	8.05	9.57	11.08	11.57	4.95	5.32	6.07	7.76	8.98	6.69	EL
DCT																						0	0.93	2.56	3.91	5.94	7.2	8.36	9.11	10.67	12.18	12.66	5.93	6.31	7.06	8.74	9.94	7.58	DCT
AVD																							0	1.66	2.99	6.85	8.09	9.23	9.98	11.44	12.9	13.35	6.86	7.23	7.99	9.67	10.87	8.52	AVD
KNS																								0	1.34	8.4	9.59	10.66	11.32	12.69	14.04	14.42	8.44	8.82	4.58	11.26	12.47	10.15	KNS
IC																									0	9.67	10.83	11.85	12.45	13.75	15.01	15.36	9.74	10.11	10.88	12.56	13.78	11.48	IC
GNT																										0	1.35	2.77	3.86	5.62	7.41	8.08	0.53	0.73	1.37	2.95	4.19	2.48	GNT
WE																											0	1.45	2.6	4.38	6.2	6.9	1.66	1.52	1.43	2.27	3.42	2.65	WE
OAK																												0	1.18	2.94	4.77	5.48	3.11	2.96	2.71	2.75	3.55	3.76	OAK
LW																													0	1.78	3.6	4.31	4.26	4.12	3.9	3.76	4.34	4.92	LW
EP																														0	1.83	2.56	6.02	5.9	5.66	5.29	5.6	6.61	EP
COL																															0	0.77	7.83	7.72	7.48	7.07	7.25	8.43	COL
APT																																0	8.5	8.4	8.19	7.8	8	9.16	APT
DOME																																	0	0.38	1.14	2.83	4.05	2.08	DOME
VC																																		0	0.75	2.45	3.66	1.75	VC
ASH																																			0	1.68	2.91	1.25	ASH
WL																																				0	1.24	1.66	WL
HAM																																					0	2.57	HAM
BNK																																						0	BNK
	NS	SS	D	MC	BH	LC	LNX	BHOG	CH	DRA	Arts	MT	NA	CVC	PCH	SPT	GS	K	I	EW	EL	DCT	AVD	KNS	IND	GNT	WE	OAK	LW	EP	COL	APT	DOME	VC	ASH	WL	HAM	BNK	

Table 1. Actual rail distances

Using Google Earth we determined the Euclidean distance between every pair of stations.

LC						0	1.9	3.4	6.1	8.1	3.2	3.7	4.3	4.7	5.2	5.7	6.1	6.8	8.2	9	10.7	12	12.8	14.7	16	6.1	7.6	9.1	10.2	12.1	13.9	14.7	6.1	6.5	7.2	8.9	10.4	8.6	LC
LNX							0	1.5	4.2	6.2	5.1	5.1	5.7	6.1	6.6	7.1	7.5	8.2	9.6	10.4	12.1	13.4	14.2	16.1	17.4	7.5	9	10.5	11.6	13.5	15.3	16.1	7.5	7.9	8.6	10.3	11.8	10	LNX
BHOG								0	2.7	4.7	6.6	6.6	7.2	7.6	8.1	8.6	9	9.7	11.1	11.9	13.6	14.9	15.7	17.6	18.9	9	10.5	12	13.1	15	16.8	17.6	9	9.4	10.1	11.8	13.3	11.5	BHOG
CH									0	2	9.3	9.3	9.9	10.3	10.8	11.3	11.7	12.4	13.8	14.6	16.3	17.6	18.4	20.3	21.6	11.7	13.2	14.7	15.8	17.7	19.5	20.3	11.7	12.1	12.8	14.5	16	14.2	CH
DRA										0	11.3	11.3	11.9	12.3	12.8	13.3	13.7	14.4	15.8	16.6	18.3	19.6	20.4	22.3	23.6	13.7	15.2	16.7	17.8	19.7	21.5	22.3	13.7	14.1	14.8	16.5	18	16.2	DRA
ARTS											0	0.5	1.1	1.5	2	2.5	2.9	3.6	5	5.8	7.5	8.8	9.6	11.5	12.8	2.9	4.4	5.9	7	8.9	10.7	11.5	2.9	3.3	4	5.7	7.2	5.4	ARTS
MT												0	0.6	1	1.5	2	2.4	3.1	4.5	5.3	7	8.3	9.1	11	12.3	2.4	3.9	5.4	6.5	8.4	10.2	11	2.4	2.8	3.5	5.2	6.7	4.9	MT
NA													0	0.6	0.9	1.4	1.8	2.5	3.9	4.7	6.4	7.7	8.5	10.4	11.7	1.8	3.3	4.8	5.9	7.8	9.6	10.4	1.8	2.2	2.9	4.6	6.1	4.3	NA
CVC														0	0.5	1	1.4	2.1	3.5	4.3	6	7.3	8.1	10	11.3	1.4	2.9	4.4	5.5	7.4	9.2	10	1.4	1.8	2.5	4.2	5.7	3.9	CVC
PCH															0	0.5	0.9	1.6	3	3.8	5.5	6.8	7.6	9.5	10.8	0.9	2.4	3.9	5	6.9	8.7	9.5	0.9	1.3	2	3.7	5.2	3.4	PCH
SPT																0	0.4	1.1	2.5	3.3	5	6.3	7.1	9	10.3	0.4	1.9	3.4	4.5	6.4	8.2	9	0.4	0.8	1.5	3.2	4.7	2.9	SPT
GS																	0	0.7	2.1	2.9	4.6	5.9	6.7	8.6	9.9	0.8	2.3	3.8	4.9	6.8	8.6	9.4	0.8	1.2	1.9	3.6	5.1	3.3	GS
K																		0	1.4	2.2	3.9	5.2	6	7.9	9.2	1.5	3	4.5	5.6	7.5	9.3	10.1	1.5	1.9	2.6	4.3	5.8	4	K
I																			0	0.8	2.5	3.8	4.6	6.5	7.8	2.9	4.4	5.9	7	8.9	10.7	11.5	2.9	3.3	4	5.7	7.2	5.4	I
EW																				0	1.7	3	3.8	5.7	7	3.7	5.2	6.7	7.8	9.7	11.5	12.3	3.7	4.1	4.8	6.5	8	6.2	EW
EL																					0	1.3	2.1	4	5.3	5.4	6.9	8.4	9.5	11.4	13.2	14	5.4	5.8	6.5	8.2	9.7	7.9	EL
DCT																						0	0.8	2.7	4	6.7	8.2	9.7	10.8	12.7	14.5	15.3	6.7	7.1	7.8	9.5	11	9.2	DCT
AVD																							0	1.9	3.2	7.5	9	10.5	11.6	13.5	15.3	16.1	7.5	7.9	8.6	10.3	11.8	10	AVD
KNS																								0	1.3	9.4	10.9	12.4	13.5	15.4	17.2	18	9.4	9.8	10.5	12.2	13.7	11.9	KNS
IC																									0	10.7	12.2	13.7	14.8	16.7	18.5	19.3	10.7	11.1	11.8	13.5	15	13.2	IC
GNT																										0	1.5	3	4.1	6	7.8	8.6	0.8	1.2	1.9	3.6	5.1	3.3	GNT
WE																											0	1.5	2.6	4.5	6.3	7.1	2.3	2.7	3.4	5.1	6.6	4.8	WE
OAK																												0	1.1	3	4.8	5.6	3.8	4.2	4.9	6.6	8.1	6.3	OAK
LW																													0	1.9	3.7	4.5	4.9	5.3	6	7.7	9.2	7.4	LW
EP																														0	1.8	2.6	6.8	7.2	7.9	9.6	11.1	9.3	EP
COL																															0	0.8	8.6	9	9.7	11.4	12.9	11.1	COL
APT																																0	9.4	9.8	10.5	12.2	13.7	11.9	APT
DOME																																	0	0.4	1.1	2.8	4.3	2.5	DOME
VC																																		0	0.7	2.4	3.9	2.1	VC
ASH																																			0	1.7	3.2	1.4	ASH
WL																																				0	1.5	3.1	WL
HAM																																					0	4.6	HAM
BNK																																						0	BNK
	NS	SS	D	MC	BH	LC	LNX	BHOG	CH	DRA	Arts	MT	NA	CVC	PCH	SPT	GS	K	I	EW	EL	DCT	AVD	KNS	IND	GNT	WE	OAK	LW	EP	COL	APT	DOME	VC	ASH	WL	HAM	BNK	

Table 2. Euclidean distances between stations

In our analysis we only consider distances between stations, and not the length of a track in a particular station. Using Google Earth, we found the Euclidean distances (in miles) between every pair of rail stations. For a map of the MARTA Subway network where the scale is Euclidean distance, see Figure 9. The sum of the Euclidean efficiencies was then computed to be 379.8169. Using rail distances provided by MARTA we calculated the actual efficiencies with total sum of 311.7036. Hence, E_{Ratio}(MARTA) $= \frac{311.7036}{379.8169} = 0.8207$.

This means that the MARTA system is roughly 82% as efficient (in terms of distance) as a system that has every station connected to every other station by a direct rail line. Thus on average, the distance between stations along the rails is roughly $1/0.8207 = 1.2185$ times the direct distance.

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