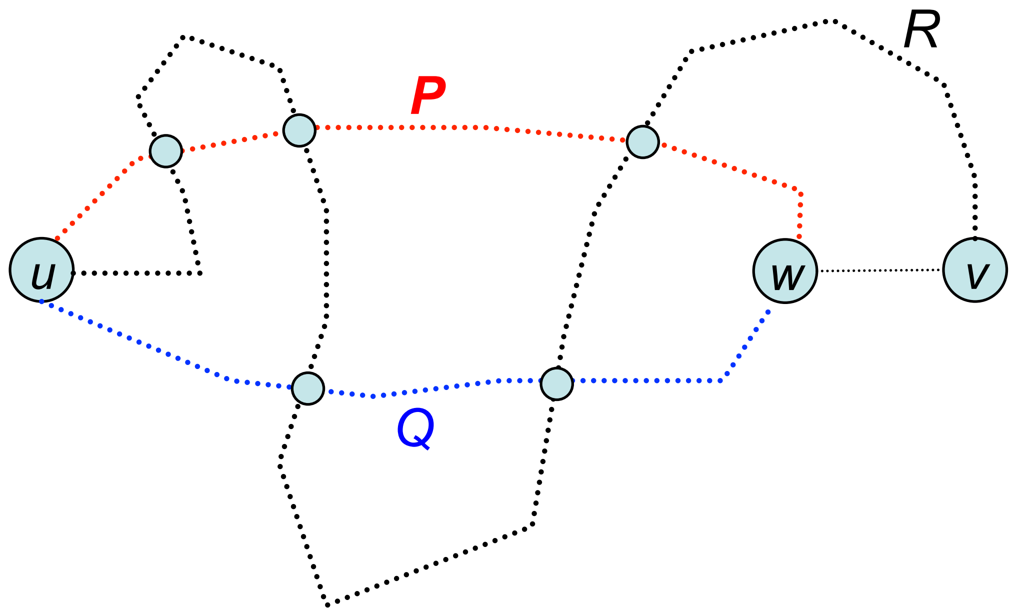
4.2 *k*-connected graphs

4.2 No Loops in Chapter 4

For all of Chapter 4, graphs have no loops. This applies to the statements and proofs of all results.

Internally Disjoint Paths

4.2.1 Definition Two paths from *u* to *v* are **internally disjoint** if they have no common internal vertex.



P, *Q* are internally disjoint *u*,*w*-paths

P,wv,v and *R* are **not** internally disjoint *u*,*v*-paths

Q,*wv,v R* are **not** internally disjoint *u*,*v*-paths

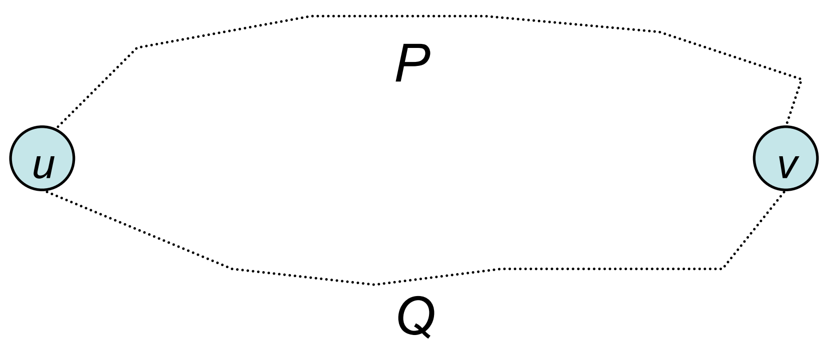
4.2 A characterization for 2-connectedness

4.2.2 Theorem. (Whitney [1932]) A graph G having at least 3 vertices is 2-connected iff for all $u,v \in V(G)$ there exist internally disjoint u,v-paths in G.

Proof:

(<= Sufficiency) Let $S = \{w\} \subseteq V(G)$. Let $u,v \in G–S$.

Let P,Q be internally disjoint u,v-paths in G:



w can be on at most one of these paths, so removing w fails to disconnect u and v.

4.2 A characterization for 2-connectedness

4.2.2 Theorem. (Whitney [1932]) A graph G having at least 3 vertices is 2-connected iff for all $u,v \in V(G)$ there exist internally disjoint u,v-paths in G.

(=> *Necessity*)

Assume G is 2-connected. Let $u,v \in V(G)$.

Induction on $d(u,v)$:

<u>Base case</u> $d(u,v)=1$

Since $d(u,v)=1$, there is an edge uv in G.

Since $\kappa'(G) \geq \kappa(G)$ and $\kappa(G) \geq 2$, $\kappa'(G) \geq 2$ is forced.

Therefore $G-uv$ is connected.

The two internally disjoint u,v-paths required are:

(1) u, uv, v and

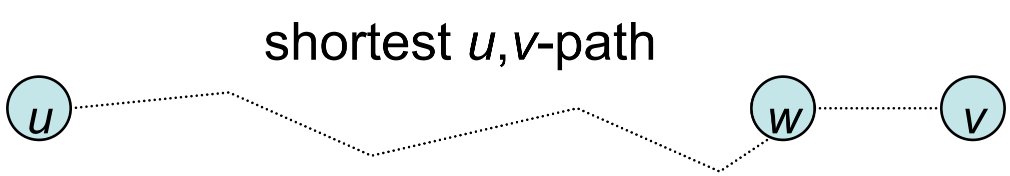
(2) A u,v-path in $G-uv$.

4.2 A characterization for 2-connectedness

4.2.2 Theorem. (Whitney [1932]) A graph G having at least 3 vertices is 2-connected iff for all $u,v \in V(G)$ there exist internally disjoint u,v-paths in G.

Induction step $d(u,v) > 1$

Let w be the vertex adjacent to v on some shortest u,v-path.



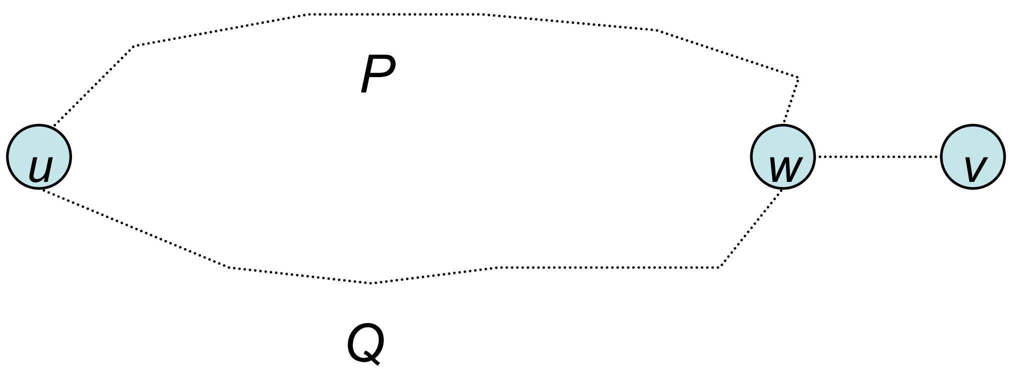
4.2 A characterization for 2-connectedness

4.2.2 Theorem. (Whitney [1932]) A graph G having at least 3 vertices is 2-connected iff for all $u,v \in V(G)$ there exist internally disjoint u,v-paths in G.

Induction step $d(u,v) > 1$

Let w be the vertex adjacent to v on some shortest u,v-path.

Since $d(u,w)=d(u,v)-1$, by induction there exist internally disjoint u,w-paths P and Q.

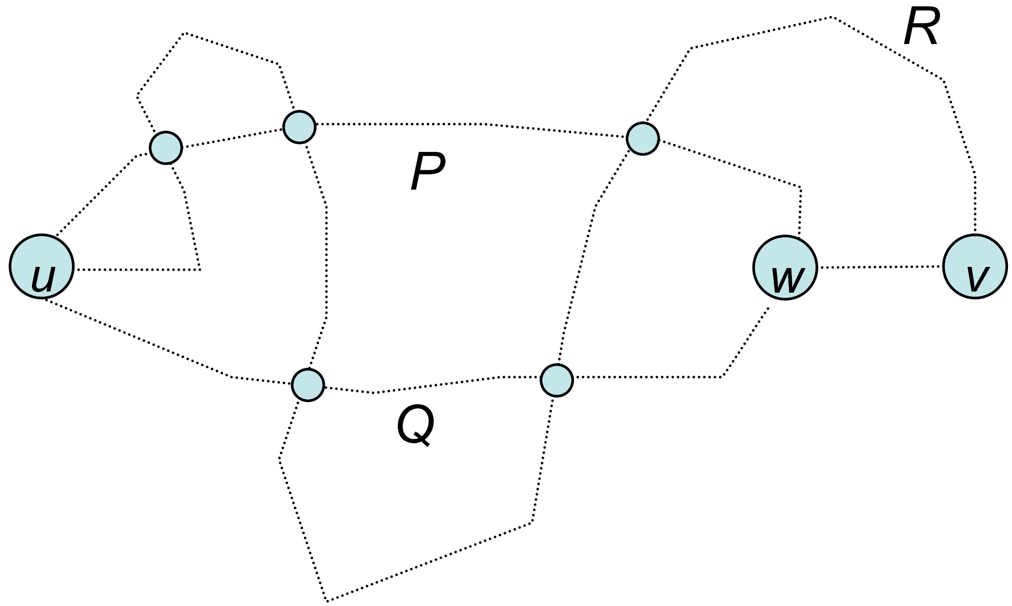


4.2 A characterization for 2-connectedness

4.2.2 Theorem. (Whitney [1932]) A graph G having at least 3 vertices is 2-connected iff for all $u,v \in V(G)$ there exist internally disjoint u,v-paths in G.

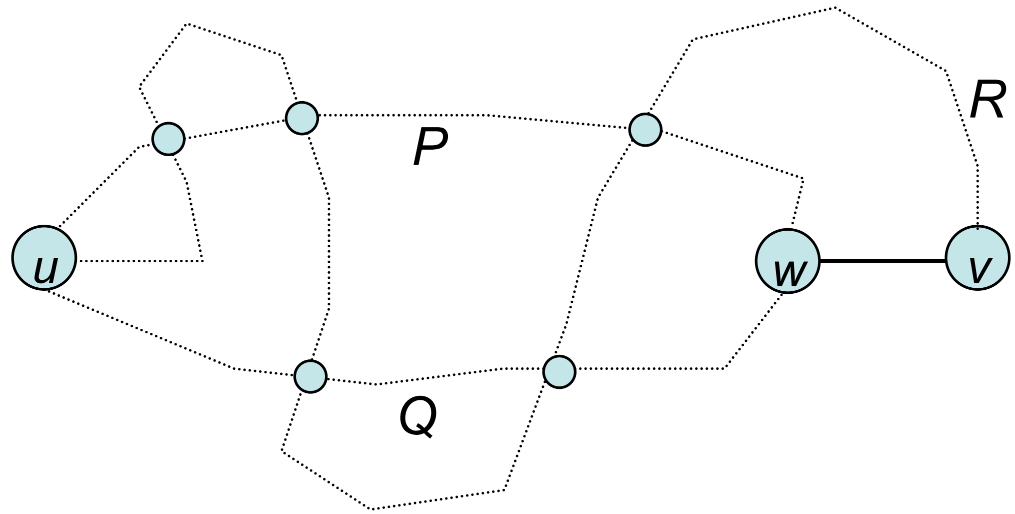
$G-w$ is connected since $\kappa(G)=2$.

Thus there is a u,v-path in $G-w$; call it R.

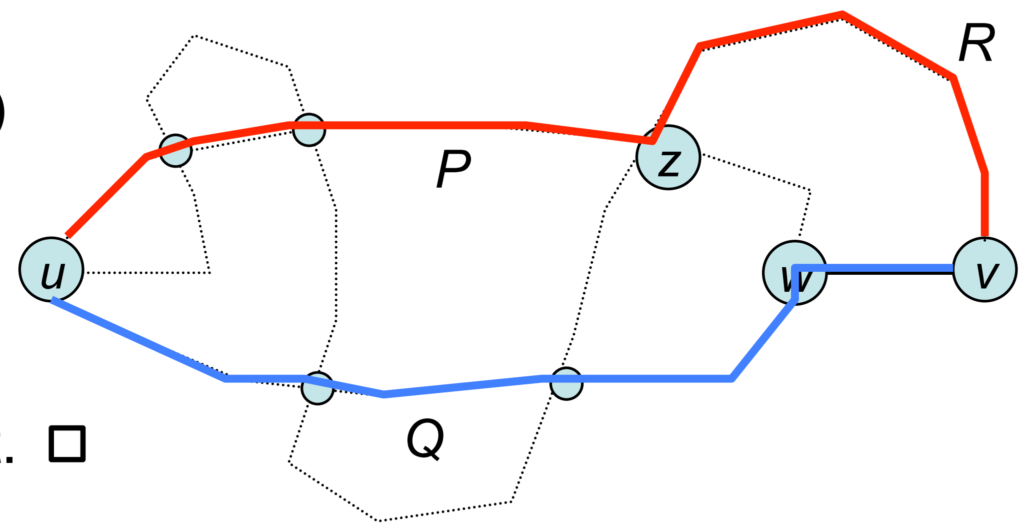


4.2 A characterization for 2-connectedness

Now look at the original *G*.
If *R* is internally disjoint from *P* (or from *Q*), we have 2 internally disjoint *u*,*v*-paths, *R* and *P* (or *Q*).



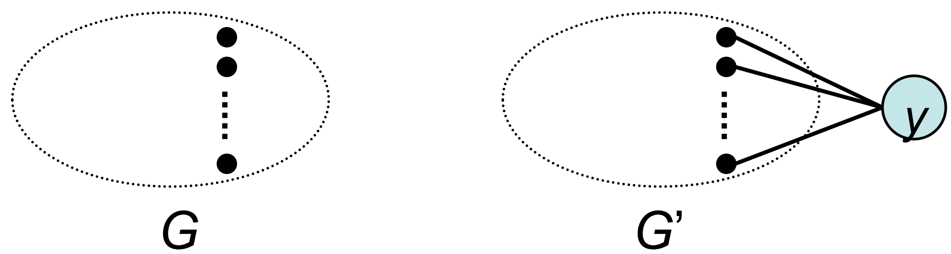
Otherwise, WLOG,
the last vertex on $R \cap (P \cup Q)$ is $z \in V(P)$. Use the paths:
(1) *P* to *z* then *R*; and
(2) *Q*.
These are internally disjoint. □



4.2 Expansion Lemma

4.2.3 Lemma. (Expansion Lemma) If G is a k-connected graph, and G' is obtained from G by adding a new vertex y with at least k neighbors in G, then G' is k-connected.

Proof:



Let S be a vertex set that:

(a) Is a vertex cut for G'; or (b) has $n(G'-S)=1$.

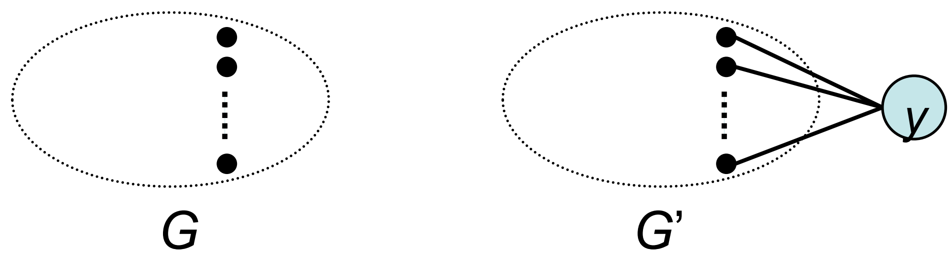
If (b) is true, then $|S \cap V(G)| \geq k$; therefore $|S| \geq k+1$.

(We only have to worry about size k vertex cuts of G'.)

4.2 Expansion Lemma

4.2.3 Lemma. (Expansion Lemma) If G is a k-connected graph, and G' is obtained from G by adding a new vertex y with at least k neighbors in G, then G' is k-connected.

Proof:



Let S be a vertex set that is a vertex cut for G'.

<u>Case 1</u> S contains y.

Then $S-\{y\}$ is a vertex cut for G with $|S-\{y\}| \geq k$, so $|S| \geq k+1$.

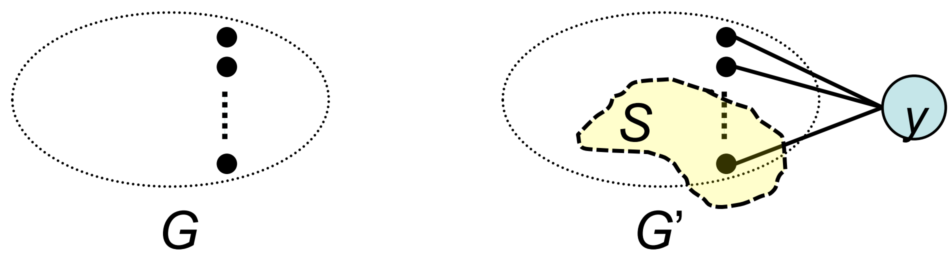
<u>Case 2</u> S does not contain y, but contains $N(y)$. Then $|S| \geq k$.

(continued next slide)

4.2 Expansion Lemma

4.2.3 Lemma. (Expansion Lemma) If *G* is a *k*-connected graph, and *G*' is obtained from G by adding a new vertex *y* with at least *k* neighbors in *G*, then *G*' is *k*-connected.

Proof:



Let *S* be a vertex set that is a vertex cut for *G*'.

<u>Case 3</u> *S* does not contain *y* and contains at most part of *N*(*y*)

Let $T = N(y) - S$ and note that $0 < |T|$.

Then *y* is in the same component of $G' - S$ as *T*, and *S* must be a vertex cut for *G*.

Therefore $|S| \geq k$.

Always a vertex cut of *G*' has size $\geq k$, so *G*' is *k*-connected. □