4.2 $k$-connected graphs

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For all of Chapter 4, graphs have no loops. This applies to the statements and proofs of all results.
4.2.1 Definition  Two paths from $u$ to $v$ are internally disjoint if they have no common internal vertex.

$P$, $Q$ are internally disjoint $u,w$-paths
$P,wv,v$ and $R$ are not internally disjoint $u,v$-paths
$Q,wv,v$ $R$ are not internally disjoint $u,v$-paths
4.2.2 Theorem. (Whitney [1932]) A graph $G$ having at least 3 vertices is 2-connected iff for all $u,v \in V(G)$ there exist internally disjoint $u,v$-paths in $G$.

Proof:

(<= Sufficiency) Let $S = \{w\} \subseteq V(G)$. Let $u,v \in G-S$. Let $P,Q$ be internally disjoint $u,v$-paths in $G$:

$w$ can be on at most one of these paths, so removing $w$ fails to disconnect $u$ and $v$. 

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$(\Rightarrow)$ Necessity

Assume $G$ is 2-connected. Let $u,v \in V(G)$.

Induction on $d(u,v)$:

**Base case** $d(u,v)=1$

Since $d(u,v)=1$, there is an edge $uv$ in $G$.

Since $\kappa'(G) \geq \kappa(G)$ and $\kappa(G) \geq 2$, $\kappa'(G) \geq 2$ is forced.

Therefore $G-uv$ is connected.

The two internally disjoint $u,v$-paths required are:

1. $u, uv, v$

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4.2.2 Theorem. (Whitney [1932]) A graph $G$ having at least 3 vertices is 2-connected iff for all $u,v \in V(G)$ there exist internally disjoint $u,v$-paths in $G$.

**Induction step** $d(u,v) > 1$

Let $w$ be the vertex adjacent to $v$ on some shortest $u,v$-path.
4.2 A characterization for 2-connectedness

4.2.2 Theorem. (Whitney [1932]) A graph \( G \) having at least 3 vertices is 2-connected iff for all \( u, v \in V(G) \) there exist internally disjoint \( u, v \)-paths in \( G \).

Induction step \( d(u, v) > 1 \)

Let \( w \) be the vertex adjacent to \( v \) on some shortest \( u, v \)-path.

Since \( d(u, w) = d(u, v) - 1 \), by induction there exist internally disjoint \( u, w \)-paths \( P \) and \( Q \).
4.2.2 Theorem. (Whitney [1932]) A graph $G$ having at least 3 vertices is 2-connected iff for all $u,v \in V(G)$ there exist internally disjoint $u,v$-paths in $G$.

$G - w$ is connected since $\kappa(G) = 2$.
Thus there is a $u,v$-path in $G - w$; call it $R$. 
4.2 A characterization for 2-connectedness

Now look at the original $G$. If $R$ is internally disjoint from $P$ (or from $Q$), we have 2 internally disjoint $u,v$-paths, $R$ and $P$ (or $Q$).

Otherwise, WLOG, the last vertex on $R \cap (P \cup Q)$ is $z \in V(P)$. Use the paths:

1. $P$ to $z$ then $R$; and
2. $Q$.

These are internally disjoint. $\square$
4.2.3 Lemma. (Expansion Lemma) If $G$ is a $k$-connected graph, and $G'$ is obtained from $G$ by adding a new vertex $y$ with at least $k$ neighbors in $G$, then $G'$ is $k$-connected.

Proof:

Let $S$ be a vertex set that:
(a) is a vertex cut for $G'$; or
(b) has $n(G' – S) = 1$.

If (b) is true, then $|S \cap V(G)| \geq k$; therefore $|S| \geq k+1$.
(We only have to worry about size $k$ vertex cuts of $G'$.)
4.2.3 Lemma. (Expansion Lemma) If $G$ is a $k$-connected graph, and $G'$ is obtained from $G$ by adding a new vertex $y$ with at least $k$ neighbors in $G$, then $G'$ is $k$-connected.

Proof:

Let $S$ be a vertex set that is a vertex cut for $G'$.

Case 1  $S$ contains $y$.
Then $S\setminus\{y\}$ is a vertex cut for $G$ with $|S\setminus\{y\}| \geq k$, so $|S| \geq k+1$.

Case 2  $S$ does not contain $y$, but contains $N(y)$. Then $|S| \geq k$.

(continued next slide)
4.2.3 Lemma. (Expansion Lemma) If $G$ is a $k$-connected graph, and $G'$ is obtained from $G$ by adding a new vertex $y$ with at least $k$ neighbors in $G$, then $G'$ is $k$-connected.

**Proof:**

Let $S$ be a vertex set that is a vertex cut for $G'$.

**Case 3**  $S$ does not contain $y$ and contains at most part of $N(y)$

Let $T = N(y) - S$ and note that $0 < |T|$.

Then $y$ is in the same component of $G' - S$ as $T$, and $S$ must be a vertex cut for $G$.

Therefore $|S| \geq k$.

Always a vertex cut of $G'$ has size $\geq k$, so $G'$ is $k$-connected. \(\square\)